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Escola de Pós-Graduação
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em Economia

da Fundação

Getúlio Vargas

Nº 703

ISSN 0104-8910

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Março de 2010

URL: <http://hdl.handle.net/10438/4266>

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Competitive equilibria in infinite-horizon
collateralized economies with default penalties/

V. F. Martins-da-Rocha, Y. Vailakis - Rio de Janeiro :
FGV,EPGE, 2010

(Ensaaios Econômicos; 703)

Inclui bibliografia.

CDD-330

Competitive equilibria in infinite-horizon collateralized economies with default penalties ^{*}

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February 18, 2010

Abstract

Araujo, Páscoa and Torres-Martínez (2002) have shown that, without imposing either debt constraints or transversality conditions, Ponzi schemes are ruled out in infinite horizon economies with default when collateral is the only mechanism that partially secures loans. Páscoa and Seghir (2009) subsequently show that Ponzi schemes may reappear if, additionally to the seizure of the collateral, there are sufficiently harsh default penalties assessed (directly in terms of utility) against the defaulters. They also claim that if default penalties are moderate then Ponzi schemes are ruled out and existence of a competitive equilibrium is ensured. The objective of this paper is two fold. First, contrary to what is claimed by Páscoa and Seghir (2009), we show that moderate default penalties do not always prevent agents to run a Ponzi scheme. Second, we provide an alternative condition on default penalties that is sufficient to rule out Ponzi schemes and ensure the existence of a competitive equilibrium.

JEL Classification: D52, D91

Keywords: Infinite horizon economies; Incomplete markets; Default penalties; Collateral; Ponzi schemes

^{*}The financial support of the GIP ANR and of the Risk Foundation (Groupama Chair) are gratefully acknowledged by V. Filipe Martins-da-Rocha. Yiannis Vailakis acknowledges the financial support of an ERC starting grant (FP7, Ideas specific program). This paper has benefited from comments by seminar participants and discussants at the University of Warwick (CRETA seminar), the University of Illinois at Urbana-Champaign (Conference in honor of Wayne Shafer), the University of Copenhagen (Workshop on Economic Theory), the XVII European Workshop on General Equilibrium Theory and the IX Conference of the Society for the Advancement of Economic Theory. We would like to thank Joseph Abdou, Aloisio Araújo, Yves Balasko, Pablo Becker, Jean-Marc Bonnisseau, Luis H. B. Braido, Andrés Carvajal, Rose-Anne Dana, Monique Florenzano, Peter Hammond, Felix Kubler, David Levine, Michael Magill, Paulo K. Monteiro, Joseph Ostroy, Mário Páscoa, Herakles Polemarchakis, Martine Quinzii, Juan Pablo Torres-Martínez, Mich Tvede, Karl Schmedders, Abdelkrim Seghir and particularly Myrna Wooders for comments and suggestions.

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1 Introduction

A central issue that arises in infinite-horizon sequential markets models is related to the existence of the so-called Ponzi schemes. In the absence of a terminal date agents will seek to renew their credit by successively postponing the repayment of their debts until infinite. This problem has no counterpart in finite horizon economies since the requirement that agents must balance their debts at the terminal date implies limits on debt at earlier dates. The existence of such schemes causes agents' decision problem to have no solution even in cases where the system of prices does not offer (local) arbitrage opportunities. Therefore, for an equilibrium to exist when time extends to infinite, one must specify a mechanism that limits the rate at which agents accumulate debt, namely that avoids Ponzi schemes. Broadly speaking, the various attempts proposed in the literature to deal with the issue of Ponzi schemes can be classified in two categories.

On one side, there are papers that argue in favor of debt constraints (Kehoe and Levine (1993), Magill and Quinzii (1994), Hernández and Santos (1996), Levine and Zame (1996), Zhang (1997), Alvarez and Jermann (2000), Kehoe and Levine (2001), Levine and Zame (2002)) or transversality type conditions (Magill and Quinzii (1994), Florenzano and Gourdel (1996)). Debt constraints limit the level of debt at each node while transversality conditions limit the asymptotic behavior of debt. The common feature of the proposed models hinges on the assumption made about the enforcement of payments. Without exception all models prevent default at equilibrium.

On the other side, there are papers that try to address the issue of Ponzi schemes in environments where default may appear at equilibrium.¹ Allowing for the possibility of default necessitates to specify an explicit mechanism that enforces payments. One of the most important and widespread mechanisms of securing loans and lowering the level of default in financial markets is the use of collateral.²

Araujo et al. (2002) (see also Kubler and Schmedders (2003)) showed that, without imposing any debt constraints or transversality condition, Ponzi schemes are ruled out

¹Real economic systems permit default, at least to some extent. Nowadays, there is a consensus among economists that default is consistent with the orderly functioning of financial markets. Therefore, default should be treated as an equilibrium phenomenon that becomes a consequence of the optimizing behavior of economic agents. There is a vast literature on default that dates back to the seminal contributions of Shubik (1972), Shubik and Wilson (1977) and Dubey and Shubik (1979). Default was introduced in a general equilibrium setting by Dubey, Geanakoplos and Shubik (1990) and Zame (1993). Modern theoretical contributions on default include among others, Dubey, Geanakoplos and Zame (1995), Kiyotaki and Moore (1997), Geanakoplos (1997), Geanakoplos and Zame (2002), Araujo et al. (2002), Kubler and Schmedders (2003), Dubey, Geanakoplos and Shubik (2005), Páscoa and Seghir (2009), Revil and Torres-Martínez (2010), Fostel and Geanakoplos (2008). Recently, Chatterjee, Corbae, Nakajima and RíEios-Rull (2007) and Livshits, MacGee and Tertilt (2007) have constructed models with incomplete markets and default, calibrated them to data, and used them to address policy issues.

²Collateral-using activities have expanded rapidly in recent years. Financial institutions extensively employ collateral in lending, in securities trading and derivative markets and in payment and settlement systems. Central banks generally require collateral in their credit operations. Common examples of collateralized lending are home mortgages, margin purchases of securities, overnight repurchase agreements and pawn shop loans.

in economies where collateral is the only mechanism that enforces agents to pay their debts. The intuition behind their result is as follows. Combining short-sales with the purchase of collateral constitutes a joint operation that yields non-negative returns. By non-arbitrage, at equilibrium, the price of the collateral exceeds the price of the asset, implying that collateral costs exceed the value of loans. Therefore, it becomes impossible to pay a previous debt by issuing new debt.

In most economic systems, collateral is not the only mean of securing loans: the default option usually entails additional economic consequences.³ A possible reason is that the effectiveness of collateral is rather limited in the presence of large negative shocks in the value of collateral guarantees. One approach to model additional enforcement mechanisms is to introduce linear utility penalties (see Zame (1993), Dubey et al. (2005) and the literature cited therein). Contrary to collateral constraints that we observe in practice, one may argue that it is hard to evaluate and measure utility penalties. Following Zame (1993) one may interpret default penalties as the consequences directly assessed in terms of utility of a non-modeled economic punishment.

A surprising result found by Páscoa and Seghir (2009) is that the introduction of default penalties in the model of Araujo et al. (2002) may induce payments besides the value of the collateral and lead to the reappearance of Ponzi schemes. The intuition is as follows. When penalties are severe, agents have incentives to pay more than the value of the depreciated collateral. In this case, the joint operation of combining short sales with the purchase of collateral no longer yields nonnegative returns. Therefore, loans exceed collateral costs and the possibility of running Ponzi schemes reappears.

One may think that the reappearance of Ponzi schemes is related to the particular additional enforcement mechanism (linear utility penalties) Páscoa and Seghir (2009) have considered. However, Revil and Torres-Martínez (2010) showed that any effective additional enforcement mechanism implies the non-existence of physically feasible individuals' optimal plans.⁴ That is, any effective additional enforcement mechanism gives rise to Ponzi schemes in infinite horizon collateralized economies. Hence, it is the effectiveness of the mechanism that brings the main result, not the mechanism per se.

Páscoa and Seghir (2009) claimed that collateral still avoids Ponzi schemes provided that default penalties are moderate, in the sense that the penalty associated with the maximal default for a physically feasible plan is less than the utility from consuming the current endowment. Their claim appears to be intuitive. If default penalties are moderate, then default does not hurt much since the utility from consuming the current endowment always compensates the disutility suffered from defaulting. In other words, moderate default penalties are not effective in the long run, in the sense that they do not induce payments besides the value of the collateral.

The contribution of this paper is two fold. Our first result shows that, contrary to

³For instance, if an agent files for bankruptcy under Chapter 7 of the U.S. bankruptcy code, the following things may happen (see Chatterjee et al. (2007)): (1) he is not allowed to save and his existing savings will be completely garnished; (2) he has to pay a proportion of the current income as cost of filing for bankruptcy; (3) a proportion of his current labor income is garnished; (4) his credit history turns bad and he is excluded from the loan market.

⁴An enforcement mechanism is said effective if it entails payments besides the value of the collateral.

what is claimed by Páscoa and Seghir (2009), moderate default penalties do not always rule out Ponzi schemes. We provide a specific example showing that moderate default penalties can be effective and induce agents to pay fully their debt at every period. This fact induces agents to run a Ponzi scheme.

This finding leads us to question whether there are default penalties that preclude agents to run Ponzi schemes, and therefore are compatible with equilibrium existence in collateralized economies. We provide an affirmative answer to this question by characterizing a family of default penalties that are not effective in the long run. In particular, we provide a sufficient condition on default penalties (expressed in terms of the primitives of the economy) that precludes Ponzi schemes: the marginal utility of consuming the collateral should be eventually larger than the marginal default penalty. It is this sufficient condition that captures the intuition behind the effectiveness of default penalties conjectured by Páscoa and Seghir (2009).⁵

The paper is structured as follows. In Section 2 we set out the model, introduce notation and the associated equilibrium concept. Section 3 contains the assumptions imposed on the characteristics of the economy. Section 4 shows that Ponzi schemes are not always ruled out in the presence of moderate default penalties. In Section 5 we present an alternative (to moderate default penalties) sufficient condition on default penalties and show that it is compatible with the existence of equilibrium. Some technical results are postponed to the appendix.

2 The Model

The model is essentially the one developed by Araujo et al. (2002) and extended by Páscoa and Seghir (2009) to allow for the possibility of linear default penalties. We consider a stochastic economy \mathcal{E} with an infinite horizon.

2.1 Uncertainty and time

Let $\mathcal{T} = \{0, 1, \dots, t, \dots\}$ denote the set of time periods and let S be a (infinite) set of states of nature. The available information at period $t \in \mathcal{T}$ is the same for each agent and is described by a finite partition P_t of S . Information is revealed along time, i.e., the partition P_{t+1} is finer than P_t for every t . Every pair (t, σ) where σ is a set in P_t is called a node. The set of all nodes is denoted by D and we call this set the event tree. We assume that there is no information at $t = 0$ and we denote by $\xi_0 = (0, S)$ the initial node. If $\xi = (t, \sigma)$ belongs to the event tree, then t is denoted by $t(\xi)$. We say that $\xi' = (t', \sigma')$ is a successor of $\xi = (t, \sigma)$ if $t' \geq t$ and $\sigma' \subset \sigma$; we use the notation $\xi' \geq \xi$. We denote by ξ^+ the set of immediate successors defined by

$$\xi^+ = \{\xi' \in D : t(\xi') = t(\xi) + 1\}.$$

⁵Our sufficient condition is in sharp contrast with the one proposed by Páscoa and Seghir (2009). Their definition of moderate default penalties involves the comparison of utility and disutility levels (disutility from defaulting and utility of consuming the initial endowment), while our characterization of the family of default penalties involves the comparison of marginal utility and disutility.

Because P_t is finer than P_{t-1} for every $t > 0$, there is a unique node ξ^- in D such that ξ is an immediate successor of ξ^- . Given a period $t \in \mathcal{T}$ we denote by D_t the set of nodes at period t , i.e., $D_t = \{\xi \in D : t(\xi) = t\}$. The set of nodes up to period t is denoted by D^t , i.e., $D^t = \{\xi \in D : t(\xi) \leq t\}$.

2.2 Agents and commodities

There exists a finite set L of commodities available for trade at every node $\xi \in D$. We interpret $x(\xi) \in \mathbb{R}_+^L$ as a claim to consumption at node ξ . We also write $\mathbf{1}_{\{\ell\}} \in \mathbb{R}_+^L$ for the commodity bundle consisting of one unit of commodity $\ell \in L$ and nothing else. We allow for some commodities to be *non-perishable*, that is, we allow for storable and durable goods. Transformation of commodities is represented by a family $(Y(\xi))_{\xi \in D}$ of linear functionals $Y(\xi)$ from \mathbb{R}_+^L to \mathbb{R}_+^L . The bundle $Y(\xi)z(\xi^-)$ represents what is obtained at node ξ if the bundle $z(\xi^-) \in \mathbb{R}_+^L$ is purchased at node ξ^- . We say that the commodity ℓ is perishable at node ξ^- if $Y(\xi)\mathbf{1}_{\{\ell\}}$ is the zero vector in \mathbb{R}_+^L . Otherwise, we say that the good ℓ is non-perishable. At each node there are spot markets for trading every commodity. We let $p = (p(\xi))_{\xi \in D}$ denote the spot price process where $p(\xi) = (p(\xi, \ell))_{\ell \in L} \in \mathbb{R}_+^L$ is the price vector at node ξ .

There is a finite set I of infinitely lived agents. Each agent $i \in I$ is characterized by an endowment process $\omega^i = (\omega^i(\xi))_{\xi \in D}$ where $\omega^i(\xi) = (\omega^i(\xi, \ell))_{\ell \in L} \in \mathbb{R}_+^L$ denotes the endowment available at node ξ . Each agent chooses a consumption process $x = (x(\xi))_{\xi \in D}$ where $x(\xi) \in \mathbb{R}_+^L$. We denote by X the set of consumption processes. The utility function $U^i : X \rightarrow [0, +\infty]$ is assumed to be additively separable, i.e.,

$$U^i(x) = \sum_{\xi \in D} u^i(\xi, x(\xi))$$

where $u^i : D \times \mathbb{R}_+^L \rightarrow [0, \infty)$.

Remark 2.1. As in Araujo et al. (2002) and Páscoa and Seghir (2009), we allow $U^i(x)$ to be infinite for some consumption process x in X .

2.3 Assets and collateral

There is a finite set J of short-lived real financial assets available for trade at each node. For each asset j , the associated return at node ξ is denoted by $A(\xi, j) \in \mathbb{R}_+^L$. We let $q = (q(\xi))_{\xi \in D}$ be the asset price process where $q(\xi) = (q(\xi, j))_{j \in J} \in \mathbb{R}_+^J$ represents the asset price vector at node ξ . We denote by $\theta^i(\xi) \in \mathbb{R}_+^J$ the vector of purchases and by $\varphi^i(\xi) \in \mathbb{R}_+^J$ the vector of short-sales at each node ξ .

Following Araujo et al. (2002) and Páscoa and Seghir (2009) (see also Geanakoplos (1997) and Geanakoplos and Zame (2002)), assets are collateralized in the sense that for every unit of asset j sold at a node ξ , agents should buy a collateral $C(\xi, j) \in \mathbb{R}_+^L$ that protects lenders in case of default. Implicitly we assume that payments can be enforced through the seizure of the collateral. At a node ξ , agent i should deliver the

promise $V(p, \xi)\varphi^i(\xi^-)$ where

$$V(p, \xi) = (V(p, \xi, j))_{j \in J} \quad \text{and} \quad V(p, \xi, j) = p(\xi)A(\xi, j).$$

However, agent i may decide to default and choose a delivery $d^i(\xi, j)$ in units of account. Denote by $d^i(\xi) = (d^i(\xi, j))_{j \in J}$ the vector of asset deliveries at node ξ . Since the collateral can be seized, this delivery must satisfy

$$d^i(\xi, j) \geq D(p, \xi, j)\varphi^i(\xi^-, j)$$

where

$$D(p, \xi, j) = \min\{p(\xi)A(\xi, j), p(\xi)Y(\xi)C(\xi^-, j)\}.$$

Following Dubey et al. (2005) and Páscoa and Seghir (2009), we assume that agent i feels a disutility $\lambda^i(\xi, j) \in [0, +\infty]$ from defaulting. If an agent defaults at node ξ , then he suffers at $t = 0$, the disutility

$$\sum_{j \in J} \lambda^i(\xi, j) \frac{[V(p, \xi, j)\varphi^i(\xi^-, j) - d^i(\xi, j)]^+}{p(\xi)v(\xi)}.$$

where $v(\xi) \in \mathbb{R}_{++}^L$ is exogenously specified. In that case, agent i may have an incentive to deliver more than the minimum between his debt and the depreciated value of his collateral, i.e., we may have $d^i(\xi, j) > D(p, \xi, j)\varphi^i(\xi^-, j)$. Assets are thought as pools, i.e., at each node ξ there is a vector $\kappa(\xi) = (\kappa(\xi, j))_{j \in J}$ of delivery rates that summarizes all different sellers's deliveries. Each asset j delivers to lenders the fraction $V(\kappa, p, \xi, j)$ per unit of asset purchased defined by

$$V(\kappa, p, \xi, j) = \kappa(\xi, j)V(p, \xi, j) + (1 - \kappa(\xi, j))D(p, \xi, j).$$

2.4 Budget constraints

We let A be the space of adapted processes $a = (a(\xi))_{\xi \in D}$ with

$$a(\xi) = (x(\xi), \theta(\xi), \varphi(\xi), d(\xi))$$

where⁶

$$x(\xi) \in \mathbb{R}_+^L, \quad \theta(\xi) \in \mathbb{R}_+^J, \quad \varphi(\xi) \in \mathbb{R}_+^J, \quad d(\xi) \in \mathbb{R}_+^J.$$

In each decision node $\xi \in D$, agent i 's choice $a^i = (x^i, \theta^i, \varphi^i, d^i) \in A$ must satisfy the following constraints:

⁶By convention we let

$$a(\xi_0^-) = (x(\xi_0^-), \theta(\xi_0^-), \varphi(\xi_0^-), d(\xi_0^-)) = (0, 0, 0, 0).$$

(a) solvency constraint:

$$p(\xi)x^i(\xi) + \sum_{j \in J} d^i(\xi, j) + q(\xi)\theta^i(\xi) \leq p(\xi)[\omega^i(\xi) + Y(\xi)x^i(\xi^-)] + V(\kappa, p, \xi)\theta^i(\xi^-) + q(\xi)\varphi^i(\xi); \quad (2.1)$$

(b) collateral requirement:

$$C(\xi)\varphi^i(\xi) \leq x^i(\xi); \quad (2.2)$$

(c) minimum delivery:

$$\forall j \in J, \quad D(p, \xi, j)\varphi^i(\xi^-, j) \leq d^i(\xi, j). \quad (2.3)$$

2.5 The payoff function

Assume that $\pi = (p, q, \kappa)$ is a process of prices and delivery rates. Consider that agent i has chosen the plan $a = (x, \theta, \varphi, d) \in A$. He gets the utility $U^i(x) \in [0, \infty]$ defined by

$$U^i(x) = \sum_{\xi \in D} u^i(\xi, x(\xi))$$

but he suffers the disutility $W^i(p, a) \in [0, \infty]$ defined by

$$W^i(p, a) = \sum_{\xi > \xi_0} \sum_{j \in J} \lambda^i(\xi, j) \frac{[V(p, \xi, j)\varphi(\xi^-, j) - d(\xi, j)]^+}{p(\xi)v(\xi)}.$$

We would like to define the payoff $\Pi^i(p, a)$ of the plan a as the following difference

$$\Pi^i(p, a) = U^i(x) - W^i(p, a).$$

Unfortunately, $\Pi^i(p, a)$ may not be well-defined if both $U^i(x)$ and $W^i(p, a)$ are infinite.⁷ We propose to consider the binary relation $\succ_{i,p}$ defined on A by

$$\tilde{a} \succ_{p,i} a \iff \exists \varepsilon > 0, \quad \exists T \in \mathbb{N}, \quad \forall t \geq T, \quad \Pi^{i,t}(p, \tilde{a}) \geq \Pi^{i,t}(p, a) + \varepsilon$$

where

$$\Pi^{i,t}(p, a) = U^{i,t}(x) - W^{i,t}(p, a), \quad U^{i,t}(x) = \sum_{\xi \in D^t} u^i(\xi, x(\xi))$$

and

$$W^{i,t}(p, a) = \sum_{\xi \in D^t \setminus \{\xi_0\}} \sum_{j \in J} \lambda^i(\xi, j) \frac{[V(p, \xi, j)\varphi(\xi^-, j) - d(\xi, j)]^+}{p(\xi)v(\xi)}.$$

Observe that if $\Pi^i(p, \tilde{a})$ and $\Pi^i(p, a)$ exist in \mathbb{R} then

$$\tilde{a} \succ_{p,i} a \iff \Pi^i(p, \tilde{a}) > \Pi^i(p, a).$$

The set $\text{Pref}^i(p, a)$ of plans strictly preferred to plan a by agent i is defined by

$$\text{Pref}^i(p, a) = \{\tilde{a} \in A : \tilde{a} \succ_{i,p} a\}.$$

⁷This issue is ignored by Páscoa and Seghir (2009).

2.6 The equilibrium concept

We denote by Ξ the set of prices and delivery rates (p, q, κ) satisfying

$$\forall \xi \in D, \quad p(\xi) \in \mathbb{R}_{++}^L, \quad q(\xi) \in \mathbb{R}_+^J, \quad \kappa(\xi) \in [0, 1]^J \quad (2.4)$$

and

$$\sum_{\ell \in L} p(\xi, \ell) + \sum_{j \in J} q(\xi, j) = 1.$$

We denote by $\text{cl}\Xi$ the closure of Ξ under the weak topology.⁸

Given a process (p, q, κ) of commodity prices, asset prices and delivery rates, we denote by $B^i(p, q, \kappa)$ the set of plans $a = (x, \theta, \varphi, d) \in A$ satisfying constraints (2.1), (2.2) and (2.3). The demand $d^i(p, q, \kappa)$ is defined by

$$d^i(p, q, \kappa) = \{a \in B^i(p, q, \kappa) : \text{Pref}^i(p, a) \cap B^i(p, q, \kappa) = \emptyset\}.$$

Definition 2.1. A competitive equilibrium for the economy \mathcal{E} is a family of prices and delivery rates $(p, q, \kappa) \in \Xi$ and an allocation $\mathbf{a} = (a^i)_{i \in I}$ with $a^i \in A$ such that

(a) for every agent i , the plan a^i is optimal, i.e.,

$$a^i \in d^i(p, q, \kappa);$$

(b) commodity markets clear at every node, i.e.,

$$\sum_{i \in I} x^i(\xi_0) = \sum_{i \in I} \omega^i(\xi_0) \quad (2.5)$$

and for all $\xi \neq \xi_0$,

$$\sum_{i \in I} x^i(\xi) = \sum_{i \in I} [\omega^i(\xi) + Y(\xi)x^i(\xi^-)]; \quad (2.6)$$

(c) asset markets clear at every node, i.e., for all $\xi \in D$,

$$\sum_{i \in I} \theta^i(\xi) = \sum_{i \in I} \varphi^i(\xi); \quad (2.7)$$

(d) deliveries match at every node, i.e., for all $\xi \neq \xi_0$ and all $j \in J$,

$$\sum_{i \in I} V(\kappa, p, \xi, j) \theta^i(\xi^-, j) = \sum_{i \in I} d^i(\xi, j). \quad (2.8)$$

The set of allocations $\mathbf{a} = (a^i)_{i \in I}$ in A satisfying the market clearing conditions (b) and (c) is denoted by F . Each allocation in F is called physically feasible. A plan $a^i \in A$ is called physically feasible if there exists a physically feasible allocation \mathbf{b} such that $a^i = b^i$. The set of physically feasible plans is denoted by F^i . We denote by $\text{Eq}(\mathcal{E})$ the set of competitive equilibria for the economy \mathcal{E} .

⁸The process (p, q, κ) belongs to $\text{cl}\Xi$ if the condition $p(\xi) \in \mathbb{R}_{++}^L$ in (2.4) is replaced by $p(\xi) \in \mathbb{R}_+^L$.

2.7 Notations

Along the paper we will use repeatedly the following notations. For each period t , we denote by A^t the set of plans $a \in A$ such that $a(\xi) = (0, 0, 0, 0)$ for each ξ satisfying $t(\xi) > t$. If a is a plan in A and t is a period, we denote by $a\mathbf{1}_{[0,t]}$ the plan in A^t which coincides with a for every node $\xi \in D^t$. We denote by B^t the set of plans a in A^t satisfying $\varphi(\xi) = 0$ for any node $\xi \in D_t$.

3 Assumptions

For each agent i , we denote by $\Omega^i = (\Omega^i(\xi))_{\xi \in D}$ the process of accumulated endowments, defined recursively by

$$\Omega^i(\xi_0) = \omega^i(\xi_0) \quad \text{and} \quad \forall \xi > \xi_0, \quad \Omega^i(\xi) = Y(\xi)\Omega^i(\xi^-) + \omega^i(\xi).$$

The process $\sum_{i \in I} \Omega^i$ of accumulated aggregate endowments is denoted by Ω . This section describes the assumptions imposed on the characteristics of the economy. It should be clear that these assumptions always hold throughout the paper.

Assumption 3.1 (Agents). For every agent i ,

(H.1) the process of accumulated endowments is strictly positive and uniformly bounded from above, i.e.,

$$\exists \bar{\Omega}^i \in \mathbb{R}_{++}^L, \quad \forall \xi \in D, \quad \Omega^i(\xi) \in \mathbb{R}_{++}^L \quad \text{and} \quad \Omega^i(\xi) \leq \bar{\Omega}^i;$$

(H.2) for every node ξ , the utility function $u^i(\xi, \cdot)$ is concave, continuous and strictly increasing,⁹ with $u^i(\xi, 0) = 0$;

(H.3) the infinite sum $U^i(\Omega)$ is finite.

Assumption 3.2 (Financial assets). For every asset j and node ξ , the collateral $C(\xi, j)$ is not zero.

Remark 3.1. Assumptions (3.1) and (3.2) are classical in the literature of infinite horizon models with collateral requirements (see e.g., Araujo et al. (2002) and Páscoa and Seghir (2009)). Observe that Assumptions (H.2) and (H.3) imply that the function U^i is weakly continuous when restricted to the order interval $[0, \Omega]$.

We recall a particular set up of our framework that has been used often in the literature.

Definition 3.1. The economy \mathcal{E} is said standard if Assumptions (3.1) and (3.2) are satisfied and if for each agent i , there exists

⁹We impose that the function $u^i(\xi, \cdot)$ is strictly increasing to simplify the exposition. This condition can be weakened as follows: for every ξ the function $u^i(\xi, \cdot)$ is non-decreasing and there exists a commodity ℓ that is strictly desirable in the sense that for every pair x, y in \mathbb{R}_+^L , we have $u^i(\xi, x + y) > u^i(\xi, x)$ provided that $y(\ell) > 0$.

(S.1) a discount factor $\beta_i \in (0, 1)$;

(S.2) a sequence $(P_t^i)_{t \geq 1}$ of beliefs about nodes at period t represented by a probability $P_t^i \in \text{Prob}(D_t)$;

(S.3) an instantaneous felicity function $v^i : D \times \mathbb{R}_+^L \rightarrow [0, \infty)$;

(S.4) an instantaneous default penalty $\mu^i(\xi, j) \in (0, \infty)$ for each node $\xi > \xi_0$;

such that for each node $\xi \in D$,

$$u^i(\xi, \cdot) = [\beta_i]^{t(\xi)} P_{t(\xi)}^i(\xi) v^i(\xi, \cdot)$$

for each $j \in J$,

$$\lambda^i(\xi, j) = [\beta_i]^{t(\xi)} P_{t(\xi)}^i(\xi) \mu^i(\xi, j)$$

and the processes $(A(\xi, j))_{\xi > \xi_0}$, $(\mu^i(\xi, j))_{\xi > \xi_0}$ and $(G(\xi, j))_{\xi \in D}$ are uniformly bounded from above, where

$$G(\xi, j) = \frac{1}{\max_{\ell \in L} C(\xi, j, \ell)}.$$

4 Ponzi schemes

When collateral repossession is the only enforcement mechanism, that is, when default penalties are assumed to be equal to zero, it was proved by Araujo et al. (2002) that an equilibrium exists. Páscoa and Seghir (2009) provide examples of collateralized economies in which Ponzi schemes reappear in the presence of harsh default penalties. The intuition behind the construction of those examples is as follows. When penalties are tough, lenders anticipate the total payment to exceed the value of the depreciated collateral guarantees. By non-arbitrage, they are willing to lend more than the current value of those guarantees, therefore loans can exceed collateral costs and Ponzi schemes may reappear.

Páscoa and Seghir (2009) conjectured that moderate default penalties may be compatible with the existence of equilibrium, that is, they may be sufficient to rule out Ponzi schemes. This conjecture appears to be very intuitive since we already know that an equilibrium exists if the default penalty is zero. If default penalties are moderate then default does not hurt much, in the sense that the utility from consuming the current endowment always compensates the disutility suffered from defaulting. In other words, moderate default penalties should not prevent agents to fully default in the long run, loosing their collateral. Ponzi schemes should be avoided, since, after a while, the joint operation of short-selling an asset and purchasing the collateral requirement should not allow to transfer wealth between periods.

In this section we show that moderate default penalties do not capture the intuition conjectured by Páscoa and Seghir (2009), namely they do not rule out Ponzi schemes. To illustrate our claim, we present a specific example of an economy with moderate default penalties that are fully effective. More precisely, we show that if a

non-trivial equilibrium does exist then agents are induced to make full repayments of debt. Consequently, by non-arbitrage, equilibrium prices are such that agents can run Ponzi schemes.

We start by introducing some notation. For each asset j and node ξ , we denote by $M(\xi, j)$ the real number

$$\min_{\ell \in L} \frac{\Omega(\xi, \ell)}{C(\xi, j, \ell)}.$$

Observe that under Assumption 3.2, we have $M(\xi, j) < \infty$. Finally, for every node $\xi \neq \xi_0$ we let¹⁰

$$H(\xi, j) = M(\xi^-, j) \sup_{p \in \Delta(L)} \frac{[pA(\xi, j) - pY(\xi)C(\xi^-, j)]^+}{pv(\xi)}.$$

The quantity $H(\xi, j)$ is the maximum amount in real terms that an agent may default on asset j if his plan is feasible. The proof of the following proposition is straightforward and omitted.

Proposition 4.1. If a in A is a plan physically feasible and (p, q, κ) in Ξ is a process of prices and delivery rates, then for each node ξ and each asset j , we have

$$\varphi(\xi, j) \leq M(\xi, j) \quad \text{and} \quad [V(p, \xi, j)\varphi(\xi^-, j) - d(\xi, j)]^+ \leq H(\xi, j).$$

Páscoa and Seghir (2009) introduced the concept of α -moderate default penalties. Fix a process $\alpha = (\alpha(\xi))_{\xi \in D}$ with $\alpha(\xi) \in (1, \infty)^J$.

Definition 4.1. Default penalties are said α -moderate with respect to utility functions, if for each agent i , for each period t , there exists $T > t$ such that

$$\sum_{\xi \in D_T} \sum_{j \in J} \lambda^i(\xi, j) \alpha(\xi, j) H(\xi, j) \leq \sum_{\xi \in D_T} u^i(\xi, \omega^i(\xi)). \quad (4.1)$$

Default penalties are said moderate with respect to utility functions, if they are α -moderate for some $\alpha \in (1, \infty)^{J \times D}$.

In other words, when default penalties are α -moderate, then sometime in the future, the penalty associated with a maximal default for a feasible plan, is less than the utility from consuming the current endowment.

Remark 4.1. Actually Páscoa and Seghir (2009) replace condition (4.1) by the following more restrictive condition:

$$\forall \xi \in D_T, \quad \sum_{j \in J} \lambda^i(\xi, j) \alpha(\xi, j) H(\xi, j) \leq u^i(\xi, \omega^i(\xi))$$

with some specific $\alpha(\xi, j) > 1$.

¹⁰The set $\Delta(L)$ is the simplex in \mathbb{R}_+^L , i.e., $\Delta(L) = \{p \in \mathbb{R}_+^L : \sum_{\ell \in L} p(\ell) = 1\}$.

Consider an economy \mathcal{E} with moderate default penalties and assume that (π, \mathbf{a}) is a competitive equilibrium where $\pi \in \Xi$ is a process $\pi = (\pi(\xi))_{\xi \in D}$ of prices and delivery rates, i.e.,

$$\pi(\xi) = (p(\xi), q(\xi), \kappa(\xi))$$

and $\mathbf{a} = (a^i)_{i \in I}$ is an allocation of plans $a^i = (a^i(\xi))_{\xi \in D}$ in A with

$$a^i(\xi) = (x^i(\xi), \theta^i(\xi), \varphi^i(\xi), d^i(\xi)).$$

In the following remark we present a property a competitive equilibrium should satisfy.

Remark 4.2. If default penalties are moderate, the process a^i satisfies the following property: for any period $t \geq 1$, there exist $\tau \geq t$ and a budget feasible τ -period process $a^{i,\tau}$ in $B^i(p, q, \kappa) \cap B^\tau$ such that

$$\Pi^i(p, a^{i,\tau}) \geq \Pi^{i,\tau-1}(p, a^i)$$

and

$$\forall \xi \in D^{\tau-1}, \quad a^{i,\tau}(\xi) = a^i(\xi).$$

Indeed, it is straightforward to check that we can choose $a^{i,\tau}$ defined as follows

$$\forall \xi \in D, \quad a^{i,\tau}(\xi) = \begin{cases} a^i(\xi) & \text{if } t(\xi) < \tau \\ (\omega^i(\xi), 0, 0, d(\xi)) & \text{if } t(\xi) = \tau \\ (0, 0, 0, 0) & \text{if } t(\xi) > \tau \end{cases}$$

where

$$d(\xi) = D(p, \xi) \varphi^i(\xi^-).$$

This property turns out to be crucial in order to get the following first order conditions.

4.1 Lagrange multipliers

Applying standard arguments,¹¹ we can prove that for each agent i there exist,

- a family of non-negative Lagrange multipliers $(\gamma^i(\xi))_{\xi \in D}$ corresponding to the sequence of budget constraints (2.1);
- for each asset j , a family of Lagrange multipliers $(\rho^i(\xi, j))_{\xi \in D}$ corresponding to the sequence of minimum delivery constraints (2.3);¹²

¹¹Although standard, the arguments are delicate. We dedicate a section (see Appendix A.2) to make all arguments transparent. One should apply Theorem A.1 by choosing $L(\xi) = L \times J \times J \times J$. Condition (L.3) in Assumption A.1 follows from (H.1) and (H.2).

¹²We let $\rho^i(\xi_0, j) = 0$ since there is no delivery at initial node ξ_0 .

- for each commodity ℓ , a family $(\chi^i(\xi, \ell))_{\xi \in D}$ of non-negative Lagrange multipliers corresponding to the sequence of collateral requirements (2.2);
- for each asset j , two families of non-negative Lagrange multipliers $(\alpha_\theta(\xi, j))_{\xi \in D}$ and $(\alpha_\varphi(\xi, j))_{\xi \in D}$ corresponding to the non-negative constraints on portfolio purchases and sales

such that for any period $\tau \geq 1$ and each finite process $a = (a(\xi))_{\xi \in D} \in \tilde{A}^\tau$,

$$\sum_{\xi \in D^{\tau+1}} \mathcal{L}^i(\xi, a(\xi), a(\xi^-)) \leq \sum_{\xi \in D} \mathcal{L}^i(\xi, a^i(\xi), a^i(\xi^-)) = \sum_{\xi \in D} \Pi^i(\xi, a^i(\xi), a^i(\xi^-)) \quad (4.2)$$

where \tilde{A} is the set of processes $(a(\xi))_{\xi \in D}$ with $a(\xi) = (x(\xi), \theta(\xi), \varphi(\xi), d(\xi))$ satisfying¹³

$$x(\xi) \in \mathbb{R}^L, \quad \theta(\xi) \in \mathbb{R}^J, \quad \varphi(\xi) \in \mathbb{R}^J \quad \text{and} \quad d(\xi) \in \mathbb{R}$$

and \tilde{A}^τ the set of processes $(a(\xi))_{\xi \in D} \in \tilde{A}$ with horizon τ , i.e., $a(\xi) = 0$ for each node ξ satisfying $t(\xi) > \tau$.

For each $\xi \in D$, the Lagrangian $\mathcal{L}^i(\xi, a(\xi), a(\xi^-))$ is defined by

$$\begin{aligned} \mathcal{L}^i(\xi, a(\xi), a(\xi^-)) &= \Pi^i(\xi, a(\xi), a(\xi^-)) \\ &+ \gamma^i(\xi) g^i(\xi, a(\xi), a(\xi^-)) \\ &+ \sum_{j \in J} \rho^i(\xi, j) h^i(\xi, j, a(\xi), a(\xi^-)) \\ &+ \sum_{\ell \in L} \chi^i(\xi, \ell) \{x(\xi, \ell) - C(\xi, \ell) \varphi(\xi)\} \\ &+ \sum_{j \in J} \alpha_\theta(\xi, j) \theta(\xi, j) + \alpha_\varphi(\xi, j) \varphi(\xi, j) \end{aligned} \quad (4.3)$$

with

$$\begin{aligned} \Pi^i(\xi, a(\xi), a(\xi^-)) &= u^i(\xi, x(\xi)) \\ &- \sum_{j \in J} \lambda^i(\xi, j) \frac{[V(p, \xi, j) \varphi(\xi^-, j) - d(\xi, j)]^+}{p(\xi) v(\xi)}, \end{aligned} \quad (4.4)$$

$$\begin{aligned} g^i(\xi, a(\xi), a(\xi^-)) &= p(\xi) \{\omega^i(\xi) + Y(\xi) x(\xi^-)\} - p(\xi) x(\xi) \\ &+ q(\xi) [\varphi(\xi) - \theta(\xi)] + V(\kappa, p, \xi) \theta(\xi^-) - \sum_{j \in J} d(\xi, j) \end{aligned} \quad (4.5)$$

and

$$h^i(\xi, j, a(\xi), a(\xi^-)) = d(\xi, j) - D(p, \xi, j) \varphi(\xi^-, j). \quad (4.6)$$

¹³By convention, we let $a(\xi_0^-) = (0, 0, 0, 0)$.

Remark 4.3. Since default penalties are moderate, we can follow the arguments in Remark 4.2 to show that condition (d) of Theorem A.1 is satisfied. Therefore, for any node $\xi \in D$, for every asset j and every commodity ℓ , we have

$$\gamma^i(\xi)g^i(\xi, a^i(\xi), a^i(\xi^-)) = 0, \quad \chi^i(\xi, \ell)\{x^i(\xi, \ell) - C(\xi, \ell)\varphi^i(\xi)\} = 0$$

together with

$$\rho^i(\xi, j)h^i(\xi, a^i(\xi), a^i(\xi^-)) = 0, \quad \alpha_\theta(\xi, j)\theta^i(\xi, j) = 0 \quad \text{and} \quad \alpha_\varphi(\xi, j)\varphi^i(\xi, j) = 0.$$

Remark 4.4. Because of the minimum delivery constraint we don't need to restrict the delivery to be non-negative, and because of the collateral requirement constraint we don't need to restrict the consumption plan to be non-negative. This is the reason why there are no Lagrange multipliers corresponding to the non-negative constraints on consumption bundles and deliveries.

It follows¹⁴ that there exist for each agent i ,

- a family of super-gradients $(\nabla u^i(\xi))_{\xi \in D}$ where $\nabla u^i(\xi)$ belongs to the super-differential $\partial u^i(\xi, x^i(\xi))$;¹⁵
- for each asset j , a family of super-gradients $(\delta^i(\xi, j))_{\xi > \xi_0}$ where $\delta^i(\xi, j)$ is a super-gradient of $\Delta \mapsto [\Delta]^+$ at $\Delta^i(\xi, j) = V(p, \xi, j)\varphi^i(\xi^-, j) - d^i(\xi, j)$,

such that

(a) first order condition for consumption: for every $\xi \in D$,

$$\nabla u^i(\xi) + \sum_{\eta \in \xi^+} \gamma^i(\eta)p(\eta)Y(\eta) + \chi^i(\xi) = \gamma^i(\xi)p(\xi); \quad (4.7)$$

(b) first order condition for asset purchases: for every $\xi \in D$,

$$\sum_{\eta \in \xi^+} \gamma^i(\eta)V(\kappa, p, \eta) \leq \gamma^i(\xi)q(\xi); \quad (4.8)$$

(c) first order condition for deliveries: for every $\xi > \xi_0$ and every $j \in J$,

$$\lambda^i(\xi, j)\delta^i(\xi, j)\frac{1}{p(\xi)v(\xi)} + \rho^i(\xi, j) = \gamma^i(\xi). \quad (4.9)$$

¹⁴See Remark A.2 in the appendix.

¹⁵Consider a finite set K , a convex subset X of \mathbb{R}^K and a concave function $f : X \subset \mathbb{R}^K \rightarrow \mathbb{R}$. The super-differential of f at $x \in X$, denoted by $\partial f(x)$, is the set of all vectors in $\zeta \in \mathbb{R}^K$ satisfying $f(y) - f(x) \leq \zeta \cdot (y - x)$ for all $y \in X$. The vectors in $\partial f(x)$ are called super-gradients of f at x .

4.2 Moderate default penalties precluding default

Our aim is to show that moderate default penalties may induce agents to optimally decide to make full payments. In order to clarify this point in a simple manner, we introduce the following list of assumptions.

Definition 4.2. The economy \mathcal{E} is said *particular* if it is standard¹⁶ and satisfies the following additional conditions:

- (P1) there is no uncertainty, i.e., for each t the set D_t reduces to a singleton $\{\xi_t\}$ and there is only one asset, i.e., $J = \{j\}$;¹⁷
- (P2) endowments are uniformly bounded away from 0, i.e., there exists a strictly positive bundle $\underline{\omega} \in \mathbb{R}_{++}^L$ such that $\omega^i(\xi_t) \geq \underline{\omega}$ for each period t and each agent i ;
- (P3) the “normalization” bundle $v(\xi_t)$ coincides with $\underline{\omega}$ for each period t ;
- (P4) for each period $t \geq 1$, there exists $b(\xi_t) > 0$ such that the promise of asset j satisfies $A(\xi_t) = b(\xi_t)\underline{\omega} + Y(\xi_t)C(\xi_{t-1})$.

Remark 4.5. When the economy is particular, the maximum amount $H(\xi_t)$ in real terms that an agent may default on asset j if his plan is feasible, satisfies the following property:

$$H(\xi_t) = M(\xi_{t-1})b(\xi_t).$$

We claim that we can choose default penalties such that they are moderate but at the same time severe enough to preclude default at equilibrium.

Proposition 4.2. Assume that the economy \mathcal{E} is particular. Fix $\alpha > 1$ and choose default penalties as follows

$$\mu^i = \alpha \frac{v^i(\bar{\Omega})}{\beta_i(1 - \beta_i)}$$

and the promises' coefficients $b(\xi_t)$ as follows

$$\forall t \geq 1, \quad b(\xi_t) = \min_{i \in I} \frac{\beta_i(1 - \beta_i)v^i(\omega^i(\xi_t))}{v^i(\bar{\Omega})} \times \frac{\max_{\ell \in L} C(\xi_{t-1}, \ell)}{\alpha \max_{\ell \in L} \bar{\Omega}(\ell)}.$$

Default penalties are moderate and if there is a competitive equilibrium for \mathcal{E} then every agent pays his debt at any period $t \geq 1$.

Proof of Proposition 4.2. It is straightforward to check that default penalties are moderate. More precisely, we have

$$\forall t \geq 1, \quad \forall i \in I, \quad \lambda^i(\xi_t)M(\xi_{t-1})b(\xi_t) \leq u^i(\xi_t, \omega^i(\xi_t)). \quad (4.10)$$

¹⁶Standard economies are defined in Section 3.

¹⁷We don't specify anymore the dependence of variables on j .

We propose now to prove that if (p, q, κ, a) is a competitive equilibrium then every agent pays his debt at any node. Fix $\tau \geq 0$, an agent i and let a in \tilde{A} be defined by

$$\forall t \geq 0, \quad a(\xi_t) = \begin{cases} a^i(\xi_t) & \text{if } t < \tau \\ (\omega^i(\xi_\tau), 0, 0, d(\xi_\tau)) & \text{if } t = \tau \\ (0, 0, 0, 0) & \text{if } t > \tau \end{cases}$$

where

$$d(\xi_\tau) = D(p, \xi_\tau) \varphi^i(\xi_{\tau-1}).$$

Since the process a belongs to \tilde{A}^τ , we can apply (4.2) to get¹⁸

$$\mathcal{L}^i(\xi_\tau, a(\xi_\tau), a(\xi_{\tau-1})) + \mathcal{L}^i(\xi_{\tau+1}, a(\xi_{\tau+1}), a(\xi_\tau)) \leq \sum_{t \geq \tau} \mathcal{L}^i(\xi_t, a^i(\xi_t), a^i(\xi_{t-1})).$$

Since $\gamma^i(\xi_t)g^i(\xi_t, a^i(\xi_t), a^i(\xi_{t-1})) = 0$ and $\rho^i(\xi_t)h^i(\xi_t, a^i(\xi_t), a^i(\xi_{t-1})) = 0$ for each period $t \geq 1$, we get

$$u^i(\xi_\tau, \omega^i(\xi_\tau)) - \lambda^i(\xi_\tau)M(\xi_{\tau-1})b(\xi_\tau) + \gamma^i(\xi_{\tau+1})p(\xi_{\tau+1})\omega^i(\xi_{\tau+1}) \leq \sum_{t \geq \tau} u^i(\xi_t, x^i(\xi_t)).$$

It follows from (4.10) that

$$\lambda^i(\xi_\tau)M(\xi_{\tau-1})b(\xi_\tau) \leq u^i(\xi_\tau, \omega^i(\xi_\tau))$$

implying that

$$\begin{aligned} \gamma^i(\xi_{\tau+1})p(\xi_{\tau+1})\omega^i(\xi_{\tau+1}) &\leq \sum_{t \geq \tau} u^i(\xi_t, \bar{\Omega}) \\ &\leq v^i(\bar{\Omega}) \sum_{t \geq \tau} [\beta_i]^t \\ &\leq [\beta_i]^\tau \frac{v^i(\bar{\Omega})}{1 - \beta_i}. \end{aligned} \tag{4.11}$$

Assume by way of contradiction that agent i is not paying his debt in asset j at date $t = \tau + 1$. The super-gradient associated to the default penalty must then satisfy $\delta^i(\xi_t) = 1$. From (4.9) we get

$$\lambda^i(\xi_t) \leq \gamma^i(\xi_t)p(\xi_t)\underline{\omega} \leq \gamma^i(\xi_t)p(\xi_t)\omega^i(\xi_t). \tag{4.12}$$

Combining (4.11) and (4.12) we get the following contradiction

$$\mu^i \leq \frac{v^i(\bar{\Omega})}{\beta_i(1 - \beta_i)}.$$

□

¹⁸Observe that for any period $t > \tau + 1$ we have

$$\mathcal{L}^i(\xi_t, a(\xi_t), a(\xi_{t-1})) = \mathcal{L}^i(\xi_t, 0, 0) \geq 0.$$

4.3 Moderate default penalties and Ponzi schemes

We claim that it is possible to choose the primitives of the economy such that, default penalties are moderate, the conditions of Proposition 4.2 are met, but a non-trivial competitive equilibrium does not exist.¹⁹ Following Páscoa and Seghir (2009), a competitive equilibrium (π, \mathbf{a}) is said *non-trivial* if for every node $\xi \in \mathcal{D}$ and each asset $j \in J$, either there is trade, i.e.,

$$\sum_{i \in I} \varphi^i(\xi, j) > 0$$

or the delivery rate $\kappa(\xi, j)$ is not zero.

Theorem 4.1. *Assume that the economy \mathcal{E} is particular. Fix $\alpha > 1$ and choose*

(a) *default penalties as follows*

$$\mu^i = \alpha \frac{v^i(\bar{\Omega})}{\beta_i(1 - \beta_i)};$$

(b) *collateral bundles $C(\xi_t)$ only in terms of a specific good $g \in L$, more precisely,*

$$\forall t \geq 0, \quad C(\xi_t) = \mathbf{1}_{\{g\}};$$

(c) *the promises' coefficients $b(\xi_t)$ as follows*

$$\forall t \geq 1, \quad b(\xi_t) = \min_{i \in I} \frac{\beta_i(1 - \beta_i)v^i(\omega^i(\xi_t))}{v^i(\bar{\Omega})} \times \frac{1}{\alpha \max_{\ell \in L} \bar{\Omega}(\ell)};$$

(d) *the utility function v^i as follows*

$$\forall x = (x(g), x(\ell)_{\ell \neq g}) \in \mathbb{R}_+^L, \quad v^i(x) = \varepsilon^i x(g) + v_{-g}^i(x(\ell)_{\ell \neq g})$$

where v_{-g}^i is differentiable, concave, strictly increasing with $v_{-g}^i(0) = 0$.

Default penalties are moderate, and choosing ε^i small enough for each i , Ponzi schemes are not ruled out, i.e., a non-trivial competitive equilibrium for \mathcal{E} cannot exist.

Proof of Theorem 4.1. Assume by way of contradiction that there exists a non-trivial competitive equilibrium $(p, q, \kappa, \mathbf{a})$ for the economy \mathcal{E} . Following Proposition 4.2, every agent i pays his debt at every period $t \geq 1$. This implies that $(p, q, \kappa, \mathbf{a})$ is also a competitive equilibrium of the economy \mathcal{E}' where agents cannot default (or where default penalties are infinite), i.e., each agent i maximizes the utility $U^i(x)$ of the plan $a \in A$ satisfying, for each period $t \geq 0$, the following budget constraint

$$p(\xi_t)x(\xi_t) + q(\xi_t)\theta(\xi_t) + V(p, \xi_t)\varphi(\xi_{t-1}) \leq q(\xi_t)\varphi(\xi_t)$$

¹⁹In that respect, we provide a counterexample to Páscoa and Seghir (2009) existence theorem.

$$+ V(p, \xi_t) \theta(\xi_{t-1}) + p(\xi_t) \{ \omega^i(\xi_t) + Y(\xi_t) x(\xi_{t-1}) \} \quad (4.13)$$

together with the collateral requirement constraint

$$C(\xi_t) \varphi(\xi_t) \leq x(\xi_t). \quad (4.14)$$

First order conditions must be satisfied, i.e., for each agent i there exist

- a family of Lagrange multipliers $(\hat{\gamma}_t^i)_{t \geq 0}$ corresponding to the sequence of budget constraints (4.13);
- for each good $\ell \in L$, a family of Lagrange multipliers $(\chi_t^i(\ell))_{t \geq 0}$ corresponding to the collateral requirement constraint (4.14);
- a family of Lagrange multipliers $(\zeta_t^i)_{t \geq 0}$ corresponding to the non-negative constraint $\varphi(\xi_t) \geq 0$;
- a family of super-gradients $(\nabla u_t^i)_{t \geq 0}$ where ∇u_t^i belongs to the super-differential $\partial u^i(\xi_t, x^i(\xi_t))$;

such that

(a) first order condition for consumption: for every $t \geq 0$,

$$\nabla u_t^i + \hat{\gamma}_{t+1}^i p(\xi_{t+1}) Y(\xi_{t+1}) + \chi_t^i = \hat{\gamma}_t^i p(\xi_t), \quad (4.15)$$

and

$$\forall \ell \in L, \quad \chi_t^i(\ell) \{ x^i(\xi_t, \ell) - C(\xi_t, \ell) \varphi^i(\xi_t) \} = 0; \quad (4.16)$$

(c) first order condition for asset purchases: for every $t \geq 0$,

$$\hat{\gamma}_{t+1}^i p(\xi_{t+1}) A(\xi_{t+1}) \leq \hat{\gamma}_t^i q(\xi_t); \quad (4.17)$$

(d) first order condition for asset short sales: for every $t \geq 0$,

$$\hat{\gamma}_t^i q(\xi_t) + \zeta_t^i = \hat{\gamma}_{t+1}^i p(\xi_{t+1}) A(\xi_{t+1}) + \chi_t^i C(\xi_t) \quad (4.18)$$

and

$$\zeta_t^i \varphi^i(\xi_t) = 0. \quad (4.19)$$

Combining (4.15) and (4.17), we get for any period $t \geq 0$ and every agent i ,

$$\begin{aligned} \hat{\gamma}_t^i \{ q(\xi_t) - p(\xi_t) C(\xi_t) \} &\geq \hat{\gamma}_{t+1}^i p(\xi_{t+1}) A(\xi_{t+1}) - \hat{\gamma}_t^i p(\xi_t) C(\xi_t) \\ &\geq b(\xi_{t+1}) \hat{\gamma}_{t+1}^i p(\xi_{t+1}) \underline{\omega} - \chi_t^i C(\xi_t) - \nabla u_t^i C(\xi_t) \\ &\geq b(\xi_{t+1}) \nabla u_{t+1}^i \underline{\omega} - \chi_t^i C(\xi_t) - \nabla u_t^i C(\xi_t). \end{aligned} \quad (4.20)$$

Since the market for commodity g at period t clears, there exists at least one agent i such that

$$x^i(\xi_t, g) > 0.$$

If $\varphi^i(\xi_t) = 0$ then it follows from (4.16) that $\chi_t^i(g) = 0$ and therefore $\chi_t^i C(\xi_t) = 0$. Consequently, we get from (4.20) that

$$\frac{\hat{\gamma}_t^i}{[\beta_i]^t} \{q(\xi_t) - p(\xi_t)C(\xi_t)\} \geq b(\xi_{t+1})\beta_i \nabla v^i(\bar{\Omega})\underline{\omega} - \varepsilon^i. \quad (4.21)$$

Observe that

$$b(\xi_t) \geq \min_{i \in I} \frac{\beta_i(1 - \beta_i)v^i(\underline{\omega})}{v^i(\bar{\Omega})} \times \frac{1}{\alpha \max_{\ell \in L} \bar{\Omega}(\ell)}.$$

It follows that it is possible to choose ε^i small enough (and independent of t), such that $q(\xi_t) - p(\xi_t)C(\xi_t) > 0$.

If $\varphi^i(\xi_t) > 0$ then it follows from (4.19) that $\zeta_t^i = 0$. Consequently, (4.15) and (4.18) imply that

$$\begin{aligned} \hat{\gamma}_t^i \{q(\xi_t) - p(\xi_t)C(\xi_t)\} &= \hat{\gamma}_{t+1}^i p(\xi_{t+1}) \{A(\xi_{t+1}) - Y(\xi_{t+1})C(\xi_t)\} - \nabla u_t^i C(\xi_t) \\ &= \hat{\gamma}_{t+1}^i b(\xi_{t+1}) p(\xi_{t+1}) \underline{\omega} - \nabla u_t^i C(\xi_t) \\ &\geq b(\xi_{t+1}) \nabla u_{t+1}^i \underline{\omega} - \nabla u_t^i C(\xi_t) \\ &\geq [\beta_i]^t \{b(\xi_{t+1})\beta_i \nabla v^i(\bar{\Omega})\underline{\omega} - \varepsilon^i\}. \end{aligned}$$

Choosing appropriately ε^i we can conclude that $q(\xi_t) > p(\xi_t)C(\xi_t)$.

We have thus proved that for each period t , we have

$$q(\xi_t) - p(\xi_t)C(\xi_t) > 0.$$

This implies that each agent can follow a Ponzi scheme which contradicts the fact that (p, q, κ, a) is a non-trivial competitive equilibrium. \square

5 Sufficient condition for existence

In this section we will restrict agents' utility functions to satisfy the following boundedness assumption.

Assumption 5.1. For every agent i there exists a process $(v^i(\xi))_{\xi \in D}$ where $v^i(\xi)$ is an upper bound for the function $u^i(\xi, \cdot)$ such that $\sum_{\xi \in D} v^i(\xi)$ is finite.

Remark 5.1. Assumption (5.1) is imposed by several authors (see for instance Kubler and Schmedders (2003), Miao (2006) and Hellwig and Lorenzoni (2009)). As in Araujo et al. (2002) our existence result follows from a truncation argument. Assumption (5.1) is used to verify (following a contradiction argument) individual optimality of the associated cluster allocations. We first show that cluster allocations dominate (in the sense that they give higher utility) any finite-horizon budgetary feasible plan. Then, using Assumption (5.1), we show that if the cluster allocation is not optimal, we can construct an alternative finite-horizon budgetary feasible plan that dominates the cluster allocation. In that way, we get the required contradiction. If the default penalty is zero (as it is the case in Araujo et al. (2002)), then Assumption (5.1) becomes superfluous.

Remark 5.2. An alternative to Assumption (5.1) is to restrict consumption processes to lie in $\ell^\infty(D)$ the space of uniformly bounded consumption processes. One may think that, following the intuition that applies to the model with full enforcement of payments presented by Levine and Zame (1996) (see footnote 8), such a restriction is innocuous.²⁰ However, in the model with linear default penalties that we consider, it is not clear whether restricting consumption processes to lie in $\ell^\infty(D)$ is innocuous. Indeed, if \tilde{a}^i is a budgetary feasible plan, we do not know if, for every $\varepsilon > 0$, we can always find an alternative budgetary feasible plan \hat{a}^i such that

$$\liminf_{t \rightarrow \infty} \{ \Pi^{i,t}(p, \hat{a}^i) - \Pi^{i,t}(p, \tilde{a}^i) \} \geq -\varepsilon$$

and the associated consumption process \hat{x}^i belongs to $\ell^\infty(D)$. The reason is that the sequences $(U^{i,t}(\tilde{x}^i))_{t \geq 0}$ and $(W^{i,t}(p, \tilde{a}^i))_{t \geq 1}$ may be simultaneously unbounded.

Before presenting our sufficient conditions, we need to introduce some notation. Given an asset $j \in J$, a node $\xi \in D$ and one of its successor $\eta \in \xi^+$, we denote by $\text{Def}(\eta, j)$ the maximum amount in real terms an agent may default at the margin (i.e., per unit of asset j sold in the predecessor node ξ) defined by²¹

$$\text{Def}(\eta, j) = \sup_{\pi \in \Delta(L)} \frac{[\pi A(\eta, j) - \pi Y(\eta) C(\xi, j)]^+}{\pi v(\eta)}$$

We denote by $\hat{\Omega}$ the process $(\hat{\Omega}(\xi))_{\xi \in D}$ of average accumulated endowments defined by

$$\forall \xi \in D, \quad \hat{\Omega}(\xi) = \frac{1}{\#I} \Omega(\xi).$$

We are looking for a condition on primitives such that the processes $p = (p(\xi))_{\xi \in D}$ and $q = (q(\xi))_{\xi \in D}$ of commodity and asset prices that are endogenously determined through supply and demand are such that Ponzi schemes do not exist. Equivalently, equilibrium prices should be such that from any node ζ , no agent can roll-over a debt at infinity. This will be the case if there exists a subsequent period $t \geq t(\zeta)$ such that for every node $\xi \in D_t$ no agent can transfer wealth from period $t+1$ to period t , i.e., if the value of collateral requirement $p(\xi)C(\xi, j)$ is greater than the amount $q(\xi, j)$ obtained from short-selling any asset j .²² We propose to show that the previous condition is guaranteed whenever the marginal utility for consuming the collateral bundle $C(\xi, j)$ is greater than the marginal penalty suffered from defaulting on asset j .

²⁰By physical feasibility and Assumption (H.1), we know that if an allocation of consumption processes $(x^i)_{i \in I}$ satisfies the market clearing conditions, then each consumption process x^i belongs to $\ell^\infty(D)$. In the model with full enforcement studied by Levine and Zame (1996), if x^i is optimal among the budgetary feasible consumption processes that belong to $\ell^\infty(D)$ then it is automatically optimal among all other budgetary feasible consumption processes. This is because, if \tilde{x}^i is budgetary feasible in Levine and Zame (1996), then for every $\varepsilon > 0$ we can find a finite-horizon budgetary feasible consumption process \hat{x}^i such that $U^i(\hat{x}^i) \geq U^i(\tilde{x}^i) - \varepsilon$.

²¹ $\Delta(L)$ is the set of prices $\pi \in \mathbb{R}_+^L$ normalized by $\sum_{\ell \in L} \pi(\ell) = 1$.

²²See Revil and Torres-Martínez (2010) for details.

Definition 5.1. We say that *the marginal utility for consuming the collateral is eventually larger than the marginal default penalty* if there exists an infinite set $\text{Bar} \subset \mathcal{T}$ of periods such that for every $t \in \text{Bar}$, for each node $\xi \in D_t$ and each asset $j \in J$, one of the following properties is satisfied:²³

(S.1) for every agent i ,

$$\forall u' \in \bar{\nabla} u^i(\xi, \hat{\Omega}(\xi)), \quad C(\xi, j)u' \geq \sum_{\eta \in \xi^+} \lambda^i(\eta, j) \text{Def}(\eta, j)$$

(S.2) there exists an agent i such that

$$\forall u' \in \bar{\nabla} u^i(\xi, \Omega(\xi)), \quad C(\xi, j)u' \geq \sum_{\eta \in \xi^+} \lambda^i(\eta, j) \text{Def}(\eta, j)$$

where for every bundle $z \in \mathbb{R}_+^L$, the set $\bar{\nabla} u^i(\xi, z)$ is the union of all super-differential $\partial u^i(\xi, y)$ when $y \in [0, z]$.²⁴

The difference between (S.1) and (S.2) stems on the bundle z where the marginal utility is defined. The lower is z the larger is $\bar{\nabla} u^i(\xi, z)$ and the weaker is the condition required. The trade-off between (S.1) and (S.2) is as follows: for (S.1) we consider a weaker condition but we require it is satisfied for every agent while for (S.2) we consider a stronger condition but it is sufficient that one agent satisfies this condition. Actually we need that the marginal utility for consuming the collateral is larger than the marginal default penalty for at least one agent i at the equilibrium consumption bundle $x^i(\xi)$. Since the equilibrium consumption is an endogenous variable we should find an exogenous lower bound for the marginal utility. Feasibility implies that for every agent i we have $x^i(\xi) \leq \Omega(\xi)$ (this explains Assumption S.2) and at least for one agent i we have $x^i(\xi) \leq \hat{\Omega}(\xi)$ (this explains Assumption S.1).

Theorem 5.1. Consider an economy satisfying Assumptions (3.1), (3.2) and (5.1). If the marginal utility for consuming the collateral is eventually larger than the marginal default penalty then a competitive equilibrium exists and Ponzi schemes are ruled out. More precisely, if $((p, q, \kappa), \alpha)$ is a competitive equilibrium then for every period $t \in \text{Bar}$ and every node $\xi \in D_t$ the collateral cost $p(\xi)C(\xi, j)$ of short-selling one unit of asset j exceeds the price $q(\xi, j)$ of the same asset.

Proof of Theorem 5.1. Fix $\tau \in \mathcal{T}$ with $\tau > 0$. Recall that A^τ denotes the set of all plans $a \in A$ such that $a(\xi) = 0$ for every node $\xi \in D$ satisfying $t(\xi) > \tau$. Recall that B^τ denotes the set of plans $a \in A^\tau$ satisfying the additional condition $\varphi(\xi) = 0$ for all node $\xi \in D$ satisfying $t(\xi) = \tau$. Given a process $(p, q, \kappa) \in \Xi$, we denote by $B^{i, \tau}(p, q, \kappa)$ the set defined by

$$B^{i, \tau}(p, q, \kappa) = B^i(p, q, \kappa) \cap B^\tau.$$

²³Every period t in Bar can be interpreted as a barrier to Ponzi schemes.

²⁴If $u^i(\xi, \cdot)$ is differentiable and the collateral bundle $C(\xi) = c(\xi)\mathbf{1}_\ell$ only depends on some commodity ℓ then $\inf\{Cu' : u' \in \bar{\nabla} u^i(\xi, z)\}$ coincides with $c(\xi)\partial u^i(\xi, z)/\partial x(\ell)$.

Definition 5.2. A competitive equilibrium for the *truncated* economy \mathcal{E}^τ is a family of prices and delivery rates $\pi = (p, q, \kappa) \in \Xi$ and an allocation $\mathbf{a} = (a^i)_{i \in I}$ with $a^i \in B^\tau$ such that

(a) for every agent i , the plan a^i is optimal, i.e.,

$$a^i \in d^{i,\tau}(p, q, \kappa) = \operatorname{argmax}\{\Pi^{i,\tau}(p, a) : a \in B^{i,\tau}(p, q, \kappa)\}; \quad (5.1)$$

(b) commodity markets clear at every node up to period τ , i.e.,

$$\sum_{i \in I} x^i(\xi_0) = \sum_{i \in I} \omega^i(\xi_0) \quad (5.2)$$

and for all $\xi \in D^\tau \setminus \{\xi_0\}$,

$$\sum_{i \in I} x^i(\xi) = \sum_{i \in I} [\omega^i(\xi) + Y(\xi)x^i(\xi^-)]; \quad (5.3)$$

(c) asset markets clear at every node up to period $\tau - 1$, i.e., for all $\xi \in D^{\tau-1}$,

$$\sum_{i \in I} \theta^i(\xi) = \sum_{i \in I} \varphi^i(\xi); \quad (5.4)$$

(d) deliveries match up to period τ , i.e., for all $\xi \in D^\tau \setminus \{\xi_0\}$ and all $j \in J$,

$$\sum_{i \in I} V(\kappa, p, \xi, j) \theta^i(\xi^-, j) = \sum_{i \in I} d^i(\xi, j). \quad (5.5)$$

Remark 5.3. If a plan \mathbf{a} belongs to B^τ , then $\Pi^{i,\tau}(p, \mathbf{a})$ and $\Pi^i(p, \mathbf{a})$ coincide for every price process p .

Remark 5.4. If (π, \mathbf{a}) is a competitive equilibrium for the truncated economy \mathcal{E}^τ , then without any loss of generality, we can assume that $q(\xi) = 0$ and $\theta(\xi) = 0$ for every terminal node $\xi \in D_\tau$.

It is claimed in Páscoa and Seghir (2009) that a competitive equilibrium for every truncated economy \mathcal{E}^τ exists, and that commodity prices are uniformly bounded away from 0. For the sake of completeness, we postpone to Appendix A.1 a simple proof of this result.

Proposition 5.1. There exists a process $m = (m(\xi))_{\xi \in D}$ of strictly positive numbers $m(\xi) > 0$ such that for every period τ , there exists a competitive equilibrium $(\pi^\tau, \mathbf{a}^\tau)$ of the truncated economy \mathcal{E}^τ satisfying $\|p^\tau(\xi)\| \geq m(\xi)$ at every node $\xi \in D^{\tau-1}$.

For each $\tau \in \mathcal{T}$ with $\tau \geq 1$, let $(\pi^\tau, \mathbf{a}^\tau)$ be a competitive equilibrium for the economy \mathcal{E}^τ where $\pi^\tau = (p^\tau, q^\tau, \kappa^\tau)$ and $\mathbf{a}^\tau = (a^{i,\tau})_{i \in I}$, satisfying $\|p^\tau(\xi)\| \geq m(\xi)$ at every node $\xi \in D^{\tau-1}$. Each process π^τ belongs to $\operatorname{cl} \Xi$ which is weakly compact as a product of compact sets. Passing to a subsequence if necessary, we can assume that the

sequence $(\pi^\tau)_{\tau \in \mathcal{T}}$ converges to a process $\pi = (p, q, \kappa)$ in $\text{cl}\Xi$. Observe that for each node $\xi \in D$, we have $\|p(\xi)\| \geq m(\xi) > 0$. In particular, for each period t and every plan $a \in A$, the payoff $\Pi^{i,t}(p, a)$ is well-defined. By feasibility at each node ξ , we get for each j

$$x^{i,\tau}(\xi) \leq \Omega(\xi), \quad \varphi^{i,\tau}(\xi, j) \leq M(\xi, j) \quad \text{and} \quad \theta^{i,\tau}(\xi, j) \leq M(\xi, j).$$

This implies that the sequence $(x^{i,\tau}(\xi), \varphi^{i,\tau}(\xi), \theta^{i,\tau}(\xi))_{\tau \in \mathcal{T}}$ is uniformly bounded. By optimality, the delivery $d^{i,\tau}(\xi, j)$ is always lower than $V(p^\tau, \xi, j)\varphi^{i,\tau}(\xi, j)$ and therefore the sequence $(d^{i,\tau}(\xi))_{\tau \in \mathcal{T}}$ is uniformly bounded. Passing to a subsequence if necessary, we can assume that for each i , the sequence $(a^{i,\tau})_{\tau \in \mathcal{T}}$ converges to a process $a^i \in A$.

We claim that (π, a) is a competitive equilibrium for the economy \mathcal{E} . It is straightforward to check that each plan a^i belongs to the budget set $B^i(p, q, \kappa)$ and that the feasibility conditions (2.5), (2.6), (2.7) and (2.8) are satisfied. The only difficulty is to prove that a^i is optimal in the budget set $B^i(p, q, \kappa)$. We split the proof in two parts. We first show that if $a \in B^i(p, q, \kappa)$ is a budget feasible plan with finite horizon then we must have $\Pi^i(p, a) \leq \Pi^i(p, a^i)$.²⁵ Then we will prove that for every period $t \in \text{Bar}$ and every node $\xi \in D_t$, the collateral cost $p(\xi)C(\xi, j)$ of short-selling one unit of asset j exceeds the price $q(\xi, j)$ of the same asset. Finally, we will show that this property is sufficient to prove that a^i is optimal among all (even with infinite horizon) budget feasible plans.

Proposition 5.2. For every agent i the plan a^i is optimal among budget feasible plans with finite horizon, i.e., for every budget feasible plan $a \in B^i(p, q, \kappa)$ if a has a finite horizon t , i.e., $a \in B^t$ then $\Pi^i(p, a) \leq \Pi^i(p, a^i)$.

The proof of Proposition 5.2 follows from standard arguments. We postpone the details to Appendix A.3. For every τ the family $((p^\tau, q^\tau, \kappa^\tau), a^\tau)$ is a competitive equilibrium of the truncated economy \mathcal{E}^τ . We use the first order conditions associated to the optimality of each plan $a^{i,\tau}$ and the assumption that the marginal utility for consuming the collateral exceeds the marginal default penalty to prove the following result.

Proposition 5.3. For every $\tau \in \mathcal{T}$ and every (barrier) period $t \in \text{Bar}$ satisfying $t < \tau$, we have

$$\forall \xi \in D_t, \quad \forall j \in J, \quad p^\tau(\xi)C(\xi, j) \geq q^\tau(\xi, j). \quad (5.6)$$

The proof is technical and the details are postponed to Appendix A.4. Passing to the limit in (5.6) we get that

$$\forall t \in \text{Bar}, \quad \forall \xi \in D_t, \quad \forall j \in J, \quad p(\xi)C(\xi, j) \geq q(\xi, j). \quad (5.7)$$

We are now ready to prove that for each agent i , the plan a^i is optimal among all budget feasible plans. Assume by way of contradiction that there exists a plan \bar{a} in the

²⁵By physical feasibility of the allocation $(a^i)_{i \in I}$ we know that $\Pi^i(p, a^i)$ is well defined.

budget set $B^i(p, q, \kappa)$, $\varepsilon > 0$ and $t^1 \in \mathbb{N}$ satisfying

$$\forall t \geq t^1, \quad \Pi^{i,t}(p, \bar{a}) > \Pi^{i,t}(p, a^i) + \varepsilon. \quad (5.8)$$

We already proved that $\lim_{t \rightarrow \infty} \Pi^{i,t}(p, a^i) = \Pi^i(p, a^i)$. It follows that there exists $t^2 \geq t^1$ such that

$$\forall t \geq t^2, \quad \Pi^{i,t}(p, a^i) + \frac{\varepsilon}{2} > \Pi^i(p, a^i). \quad (5.9)$$

Combining (5.8) and (5.9), we get that

$$\forall t \geq t^2, \quad \Pi^{i,t}(p, \bar{a}) > \Pi^i(p, a^i) + \frac{\varepsilon}{2}.$$

We consider the plan \hat{a}^t defined as follows: $\hat{a}^t(\xi)$ coincides with $\bar{a}(\xi)$ for every node ξ satisfying $t(\xi) < t$ and $\hat{a}^t(\xi) = (0, 0, 0, 0)$ for every node ξ satisfying $t(\xi) > t$. Fix now a node $\xi \in D_t$ and let

$$\hat{\theta}^t(\xi) = \hat{\varphi}^t(\xi) = 0, \quad \hat{d}^t(\xi) = d(\xi) \quad \text{and} \quad \hat{x}^t(\xi) = \bar{x}(\xi) - C(\xi)\bar{\varphi}(\xi).$$

Observe that \hat{a}^t belongs to B^t . Moreover, if t is a “barrier” period, i.e., $t \in \text{Bar}$ then \hat{a}^t is budget feasible, i.e., $\hat{a}^t \in B^i(p, q, \kappa)$. This is because $\hat{x}^t(\xi) \geq 0$ and²⁶

$$p(\xi)\bar{x}(\xi) - q(\xi)\bar{\varphi}(\xi) = p(\xi)\hat{x}^t(\xi) + [p(\xi)C(\xi) - q(\xi)]\bar{\varphi}(\xi) \geq p(\xi)\hat{x}^t(\xi).$$

Observe that

$$\Pi^{i,t}(p, \hat{a}^t) \geq \Pi^{i,t}(p, \bar{a}) - \sum_{\xi \in D_t} u^i(\xi, \bar{x}(\xi)).$$

Since the sum $\sum_{\xi \in D} u^i(\xi, \bar{x}(\xi))$ is finite, we can choose t large enough in Bar such that

$$\sum_{\xi \in D_t} u^i(\xi, \bar{x}(\xi)) \leq \frac{\varepsilon}{4}$$

implying that

$$\Pi^{i,t}(p, \hat{a}^t) \geq \Pi^i(p, a^i) + \frac{\varepsilon}{4}.$$

Since \hat{a}^t has a finite horizon and is budget feasible, this contradicts Proposition 5.2. \square

A Appendix

We collect in this appendix the proofs of some technical results.

²⁶The inequality follows from (5.7).

A.1 Proof of Proposition 5.1

We consider the following modification of the normalization of the default penalty. For every $\varepsilon > 0$ and every period τ , we let

$$W_\varepsilon^{i,\tau}(\pi, a) = \sum_{\xi \in D^\tau \setminus \{\xi_0\}} \sum_{j \in J} \lambda^i(\xi, j) \frac{[V(p, \xi, j)\varphi(\xi^-, j) - d(\xi, j)]^+}{p(\xi)v(\xi) + \varepsilon \|q(\xi)\|}$$

and

$$\Pi_\varepsilon^{i,\tau}(\pi, a) = U^{i,\tau}(x) - W_\varepsilon^{i,\tau}(\pi, a).$$

When the process π belongs to $\text{cl}\Xi$, the functions $(W_\varepsilon^{i,t})_{t \geq 1}$ are well-defined for every $\varepsilon > 0$. A pair (π, \mathbf{a}) where $\pi \in \Xi$ and $\mathbf{a} = (a^i)_{i \in I}$ is an allocation with $a^i \in B^\tau$, is said to be a competitive equilibrium of the truncated economy $\mathcal{E}_\varepsilon^\tau$ if market clearing conditions (5.2), (5.3), (5.4) and (5.5) are satisfied and the optimality condition (5.1) is replaced by

(\mathbf{a}_ε) for every agent i , the plan a^i is optimal with respect to $\Pi_\varepsilon^{i,\tau}$, i.e.,

$$a^i \in d_\varepsilon^{i,\tau}(p, q, \kappa) = \text{argmax}\{\Pi_\varepsilon^{i,\tau}(\pi, a) : a \in B^{i,\tau}(\pi)\}.$$

Observe that for every process π of prices and delivery rates in $\text{cl}\Xi$, the quantity $p(\xi)v(\xi) + \varepsilon \|q(\xi)\|$ is never 0. It is now very easy to adapt the arguments in Araujo et al. (2002) and prove that a competitive equilibrium (π, \mathbf{a}) for the truncated economy $\mathcal{E}_\varepsilon^\tau$ exists for any $\varepsilon > 0$ where $\pi \in \text{cl}\Xi$. Since utility functions are strictly increasing, we must have $p(\xi) \in \mathbb{R}_{++}^L$ for each node $\xi \in D^\tau$. We propose to exhibit an exogenous lower bound $m(\xi)$ on prices for every node ξ with $t(\xi) < \tau$. Fix a node $\xi \in D^{\tau-1}$, $\alpha > 0$ and an agent $i \in I$. Such a bound always exists when $\|q(\xi)\| < p(\xi)\bar{C}(\xi)$ where

$$\bar{C}(\xi) = \sum_{j \in J} C(\xi, j).$$

The case where $\|q(\xi)\| \geq p(\xi)\bar{C}(\xi)$ requires some attention. Let \tilde{a}_α^i be the plan in B^τ defined for every node $\zeta \in D^\tau$ by

$$\tilde{a}_\alpha^i(\zeta) = \begin{cases} a^i(\zeta) & \text{if } \zeta \notin \{\xi\} \cup \xi^+ \\ (x^i(\xi) + f(\pi, \xi)\alpha \mathbf{1}_L, \theta^i(\xi), \varphi(\xi) + \alpha \mathbf{1}_J, d^i(\xi)) & \text{if } \zeta = \xi \\ (x^i(\zeta), \theta^i(\zeta), \varphi^i(\zeta), \tilde{d}_\alpha^i(\zeta)) & \text{if } \zeta \in \xi^+ \end{cases}$$

where

$$f(\pi, \xi) = \frac{\|q(\xi)\| - p(\xi)\bar{C}(\xi)}{\|p(\xi)\|}$$

and for every j ,

$$\tilde{d}_\alpha^i(\zeta) = d^i(\zeta, j) + \alpha D(p, \zeta, j).$$

In other words, we propose to short-sell at node ξ an additional quantity $\alpha > 0$ of each asset j and to increase consumption of each commodity by $f(\pi, \xi)\alpha$ units. At each successor node $\zeta \in \xi^+$, we propose to “fully” default on additional short-sales. By doing so, at node ξ we get an additional amount of $\alpha \|q(\xi)\|$ units of accounts from short-selling. In order to satisfy the constraint imposed by the collateral requirements, we should purchase the bundle $\alpha \bar{C}(\xi)$ at node ξ . This is possible since $\|q(\xi)\| \geq p(\xi) \bar{C}(\xi)$. In other words, since $f(\pi, \xi) \geq 0$, the plan \tilde{a}_α^i belongs to the budget set $B^{i,\tau}(\pi)$ for every $\alpha > 0$. We propose to compare the payoffs of the two plans a^i and \tilde{a}_α^i .

First observe that

$$U^{i,\tau}(\tilde{x}_\alpha^i) - U^{i,\tau}(x^i) = u^i(\xi, x^i(\xi) + f(\pi, \xi)\alpha \mathbf{1}_L) - u^i(\xi, x^i(\xi)).$$

Moreover, since for each $\zeta \in \xi^+$

$$[V(p, \zeta, j)\{\varphi(\xi, j) + \alpha\} - \{d(\zeta, j) + \alpha D(p, \zeta, j)\}]^+$$

is lower than

$$[V(p, \zeta, j)\varphi(\xi, j) - d(\zeta, j)]^+ + [V(p, \zeta, j)\alpha - \alpha D(p, \zeta, j)]^+$$

we get

$$\begin{aligned} \Pi_\varepsilon^{i,\tau}(\pi, \tilde{a}_\alpha^i) - \Pi_\varepsilon^{i,\tau}(\pi, a^i) &\geq u^i(\xi, x^i(\xi) + f(\pi, \xi)\alpha \mathbf{1}_L) - u^i(\xi, x^i(\xi)) \\ &\quad - \alpha \sum_{\zeta \in \xi^+} \sum_{j \in J} \lambda^i(\zeta, j) \frac{[V(p, \zeta, j) - D(p, \zeta, j)]^+}{p(\zeta)v(\zeta) + \varepsilon \|q(\zeta)\|}. \end{aligned}$$

Let us denote by $\delta_\varepsilon^{i,\tau}$ the real number defined by

$$\delta_\varepsilon^{i,\tau} = \lim_{\alpha \rightarrow 0^+} \frac{\Pi_\varepsilon^{i,\tau}(\pi, \tilde{a}_\alpha^i) - \Pi_\varepsilon^{i,\tau}(\pi, a^i)}{\alpha},$$

and let $\nabla^+ u^i(\xi, x^i(\xi))$ be the vector in \mathbb{R}_{++}^L which its ℓ -th coordinate $\nabla_\ell^+ u^i(\xi, x^i(\xi))$ is defined by²⁷

$$\nabla_\ell^+ u^i(\xi, x^i(\xi)) = \lim_{\beta \rightarrow 0^+} \frac{u^i(\xi, x^i(\xi) + \beta \mathbf{1}_{\{\ell\}}) - u^i(\xi, x^i(\xi))}{\beta}.$$

Then

$$\begin{aligned} \delta_\varepsilon^{i,\tau} &\geq \|\nabla^+ u^i(\xi, x^i(\xi))\| f(\pi, \xi) - \sum_{\zeta \in \xi^+} \sum_{j \in J} \lambda^i(\zeta, j) \frac{[V(p, \zeta, j) - D(p, \zeta, j)]^+}{p(\zeta)v(\zeta) + \varepsilon \|q(\zeta)\|} \\ &\geq \|\nabla^+ u^i(\xi, \Omega^i(\xi))\| f(\pi, \xi) - \sum_{\zeta \in \xi^+} \sum_{j \in J} \lambda^i(\zeta, j) \frac{[V(p, \zeta, j) - D(p, \zeta, j)]^+}{p(\zeta)v(\zeta)} \\ &\geq \|\nabla^+ u^i(\xi, \Omega^i(\xi))\| f(\pi, \xi) - \sum_{\zeta \in \xi^+} \sum_{j \in J} \lambda^i(\zeta, j) \frac{H(\zeta, j)}{M(\zeta, j)}. \end{aligned}$$

²⁷The existence of $\nabla_\ell^+ u^i(\xi, x^i(\xi))$ is a consequence of the concavity of $u^i(\xi, \cdot)$. The strict monotonicity of $u^i(\xi, \cdot)$ implies that $\nabla_\ell^+ u^i(\xi, x^i(\xi))$ is strictly positive.

Therefore, if

$$f(\pi, \xi) > g(\xi) := \frac{\sum_{\zeta \in \xi^+} \sum_{j \in J} \lambda^i(\zeta, j) \frac{H(\zeta, j)}{M(\zeta, j)}}{\|\nabla^+ u^i(\xi, \Omega^i(\xi))\|}$$

then $\Pi_{\varepsilon}^{i, \tau}(\pi, \tilde{a}_\alpha^i) > \Pi_{\varepsilon}^{i, \tau}(\pi, a^i)$ for $\alpha > 0$ small enough: a contradiction. It follows that we must have

$$\frac{1 - \|p(\xi)\| - p(\xi)\bar{C}(\xi)}{\|p(\xi)\|} = f(\pi, \xi) \leq g(\xi).$$

Hence there exists $m(\xi) > 0$ depending only on the primitives of the economy \mathcal{E} such that $\|p(\xi)\| \geq m(\xi)$.

Consider now the sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ defined by

$$\forall n \in \mathbb{N}, \quad \varepsilon_n = \frac{1}{n+1}.$$

For each $n \in \mathbb{N}$, there exists an equilibrium (π_n, a_n) of the truncated economy $\mathcal{E}_{\varepsilon_n}^\tau$. Following standard arguments, there exists a process $\pi \in \text{cl}\Xi$ of prices and delivery rates and a process a of plans $a^i \in B^\tau$ such that, passing to a subsequence if necessary, the sequence $(\pi_n, a_n)_{n \in \mathbb{N}}$ converges to (π, a) . Since for each n , we have $\|p_n(\xi)\| \geq m(\xi)$ for every non-terminal node $\xi \in D^{\tau-1}$, passing to the limit, we get that $\|p(\xi)\| \geq m(\xi)$, in particular $p(\xi) > 0$ for each $\xi \in D^\tau$.²⁸ Therefore the payoff $\Pi^{i, \tau}(p, a)$ is well-defined for every plan $a \in B^\tau$. It is now standard to prove that the limit (π, a) is actually a competitive equilibrium of the truncated economy \mathcal{E}^τ .

A.2 Lagrange multipliers

For each node $\xi \in D$, we fix a finite set $L(\xi)$ of “types of action” and a subset $\Lambda(\xi)$ of $\mathbb{R}^{L(\xi)}$.²⁹ We denote by $C(L)$ the space of all processes $c = (c(\xi))_{\xi \in D}$ where $c(\xi)$ is a vector in $\Lambda(\xi)$, i.e.,

$$C(L) = \prod_{\xi \in D} \Lambda(\xi).$$

By convention, we pose $L(\xi_0^-) = \{1\}$, $\Lambda(\xi_0^-) = \{0\}$ and $c(\xi_0^-) = 0$ for any process $c \in C(L)$. For each period $T \geq 1$, we let $C^T(L)$ be the subset of $C(L)$ defined by

$$C^T(L) = \{c \in C(L) : \forall \xi \in D, \quad t(\xi) > T \Rightarrow c(\xi) = 0\}.$$

In a similar way we fix a finite set $K(\xi)$ of “constraints” on actions³⁰ and define respectively

$$C(K) = \prod_{\xi \in D} \mathbb{R}^{K(\xi)}$$

²⁸Recall that for every terminal node $\xi \in D_\tau$, we have the normalization $\|p_n(\xi)\| = 1$, implying that $p(\xi) > 0$.

²⁹In Section 4, an action $a(\xi)$ is a vector $(x(\xi), \theta(\xi), \varphi(\xi), d(\xi))$ where $x(\xi) \in \mathbb{R}^L$, $\theta \in \mathbb{R}^J$, $\varphi(\xi) \in \mathbb{R}^J$, and $d(\xi) \in \mathbb{R}^J$. For this case, we have $L(\xi) = L \times J \times J \times J$.

³⁰In Section 4, the constraints are the solvency constraint (2.1), the collateral requirement (2.2), the minimum delivery constraint (2.3) and non-negativity constraints.

and for each period $T \geq 1$,

$$C^T(K) = \{c \in C(K) : \forall \xi \in D, \quad t(\xi) > T \Rightarrow c(\xi) = 0\}.$$

For node ξ , we fix an objective function

$$f(\xi, \cdot, \cdot) : \Lambda(\xi) \times \Lambda(\xi^-) \longrightarrow \mathbb{R}$$

and a constraint function

$$g(\xi, \cdot, \cdot) : \Lambda(\xi) \times \Lambda(\xi^-) \longrightarrow \mathbb{R}^{K(\xi)}.$$

For each period $T \geq 1$ and each process $c \in C(L)$, we let

$$f^T(c) = \sum_{\xi \in D^T} f(\xi, c(\xi), c(\xi^-)).$$

When the limit exists, we denote by $f(c)$ the following sum

$$f(c) = \lim_{T \rightarrow \infty} f^T(c).$$

Given $c \in C(L)$, we denote by $g(c)$ the process in $C(K)$ defined by

$$\forall \xi \in D, \quad [g(c)](\xi) = g(\xi, c(\xi), c(\xi^-)).$$

Assumption A.1. We assume that

- (L.1) for each node ξ , the set $\Lambda(\xi)$ is convex and contains 0 and the functions $f(\xi, \cdot, \cdot)$ and $g(\xi, \cdot, \cdot)$ are concave and continuous on their domain;
- (L.2) for each node ξ , we have $f(\xi, 0, 0) = 0$ and $g(\xi, 0, 0) \geq 0$;
- (L.3) for each period $T \geq 1$, there exists a process $\hat{c} \in C^T(L)$ such that

$$f(\hat{c}) \geq 0 \quad \text{and} \quad \forall \xi \in D^T, \quad g(\xi, \hat{c}(\xi), \hat{c}(\xi^-)) \in \mathbb{R}_{++}^{K(\xi)}.$$

Applying sequentially a finite dimensional convex separation argument, we obtain the following result.

Theorem A.1. Assume that there exists $c_* \in C(L)$ such that

- (a) the process c_* satisfies the constraints $g(c_*) \geq 0$;
- (b) the sum $f(c_*)$ is well defined;
- (c) for any period $\tau \geq 1$, for every finite-time process $c \in C^\tau(L)$,

$$g(c) \geq 0 \implies f(c) \leq f(c_*).$$

The following properties hold.

1. There exists $\Psi \in C(K)$ with $\Psi(\xi) \in \mathbb{R}_+^{K(\xi)}$ such that for any period $\tau \geq 1$ and any finite process $c \in C^\tau(L)$,

$$\sum_{\xi \in D^{\tau+1}} \{f(\xi, c(\xi), c(\xi^-)) + \Psi(\xi) \cdot g(\xi, c(\xi), c(\xi^-))\} \leq f(c_*). \quad (\text{A.1})$$

2. If moreover, we have

- (d) for any period $t \geq 1$, there exist $\tau \geq t$ and a finite process $\check{c} \in C^{\tau+1}(L)$ satisfying $g(\check{c}) \geq 0$, $f(\check{c}) \geq f^\tau(c_*)$ and $c_* \mathbf{1}_{[0, \tau]} = \check{c} \mathbf{1}_{[0, \tau]}$

then

$$\forall \xi \in D, \quad \Psi(\xi) \cdot g(\xi, c_*(\xi), c_*(\xi^-)) = 0. \quad (\text{A.2})$$

3. If moreover, a process $c \in C(L)$ satisfying $g(c) \geq 0$ is such that

- (e) for any period $t \geq 1$, there exist $\tau \geq t$ and a finite process $\bar{c} \in C^{\tau+1}(L)$ satisfying $g(\bar{c}) \geq 0$, $f(\bar{c}) \geq f^\tau(c)$ and $c \mathbf{1}_{[0, \tau]} = \bar{c} \mathbf{1}_{[0, \tau]}$

then³¹

$$\liminf_{\tau \rightarrow \infty} \sum_{\xi \in D^{\tau+1}} \{f(\xi, c(\xi), c(\xi^-)) + \Psi(\xi) \cdot g(\xi, c(\xi), c(\xi^-))\} \leq f(c_*). \quad (\text{A.3})$$

In particular if $f(c)$ exists we get $f(c_*) \geq f(c)$.

Remark A.1. If we denote by $\mathcal{L}(\xi, c(\xi), c(\xi^-))$ the following expression

$$\mathcal{L}(\xi, c(\xi), c(\xi^-)) = f(\xi, c(\xi), c(\xi^-)) + \Psi(\xi) \cdot g(\xi, c(\xi), c(\xi^-))$$

then under (a)–(d) we obtain for every finite process $c \in C^\tau(L)$,

$$\sum_{\xi \in D^{\tau+1}} \mathcal{L}(\xi, c(\xi), c(\xi^-)) \leq \sum_{\xi \in D} \mathcal{L}(\xi, c_*(\xi), c_*(\xi^-)) = \sum_{\xi \in D} f(\xi, c_*(\xi), c_*(\xi^-)).$$

Remark A.2. Assume that $\Lambda(\xi)$ is an open subset of $\mathbb{R}^{L(\xi)}$. Observe that as consequence of properties (A.1) and (A.2) in Theorem A.1, we get the following variational properties:

$$\forall \xi \in D, \quad \nabla_1 \mathcal{L}(\xi, c_*(\xi), c_*(\xi^-)) + \sum_{\eta \in \xi^+} \nabla_2 \mathcal{L}(\eta, c_*(\eta), c_*(\xi)) = 0$$

where $\nabla_1 \mathcal{L}(\xi, c_*(\xi), c_*(\xi^-))$ belongs to the super-differential of the mapping $c_1 \mapsto \mathcal{L}(\xi, c_1, c_*(\xi^-))$ at $c_*(\xi)$ and $\nabla_2 \mathcal{L}(\eta, c_*(\eta), c_*(\xi))$ belongs to the super-differential of $c_2 \mapsto \mathcal{L}(\eta, c_*(\eta), c_2)$ at $c_*(\xi)$.

³¹In an earlier version of the paper, we claimed that this result was correct replacing “lim inf” by “lim” in (A.3). Actually, we don’t know if the sum converges for any process a in \tilde{A} . We would like to thank Juan Pablo Torres-Martínez for pointing this delicate issue.

Proof of Theorem A.1. Fix a period $T \geq 1$. We let A be the subset of $\mathbb{R} \times C^{T+1}(K)$ defined by

$$A = \{(\alpha, d) \in \mathbb{R} \times C^{T+1}(K) : \exists c \in C^T(L), \quad \alpha \leq f(c) - f(c_*) \quad \text{and} \quad d \leq g(c)\}$$

and we let $B = (0, \infty) \times C_+^{T+1}(K)$ where

$$C_+^{T+1}(K) = \prod_{\xi \in D^{T+1}} \mathbb{R}_+^{K(\xi)}.$$

Following Assumptions L.1–L.3 and conditions (a)–(c), the sets A and B are disjoint non-empty convex subsets of $\mathbb{R} \times C^{T+1}(K)$. It follows from the Finite Dimensional Separating Hyperplane Theorem that there exists a non-zero pair $(\mu^T, \Psi^T) \in \mathbb{R}_+ \times C_+^{T+1}(K)$ such that

$$\forall c \in C^T(L), \quad \mu^T f(c) + \sum_{\xi \in D^{T+1}} \Psi^T(\xi) \cdot g(\xi, c(\xi), c(\xi^-)) \leq \mu^T f(c_*). \quad (\text{A.4})$$

Following Assumption L.3, we can take $\mu^T = 1$ without any loss of generality.

Fix a node $\xi \in D$ and denote by τ the period $t(\xi)$. The objective is to prove that the sequence $(\Psi^T(\xi))_{T \geq 1}$ converges in \mathbb{R}_+ . Following Assumption L.3, there exists a process $\widehat{c} \in C^\tau(L)$ such that

$$f(\widehat{c}) \geq 0 \quad \text{and} \quad \forall \xi \in D^\tau, \quad \varepsilon(\xi) := g(\xi, \widehat{c}(\xi), \widehat{c}(\xi^-)) \in \mathbb{R}_{++}^{K(\xi)}.$$

Fix $T > \tau$ and observe that for any $\mu \in D$ such that $t(\mu) > \tau$ we have

$$g(\mu, \widehat{c}(\mu), \widehat{c}(\mu^-)) = g(\mu, 0, 0) \geq 0.$$

It follows from (A.4) that for all $k \in K(\xi)$,

$$\Psi^T(\xi, k) \leq \frac{f(c_*)}{\varepsilon(\xi, k)}.$$

Using a diagonal procedure and passing to a subsequence if necessary, we can prove that there exists $\Psi \in C_+(K)$ such that

$$\forall \xi \in D, \quad \Psi(\xi) = \lim_{T \rightarrow \infty} \Psi^T(\xi).$$

Now we fix a period $\tau \geq 1$ and a finite process $c \in C^\tau(L)$. For each $T > \tau$, it follows from (A.4) and Assumption (L.2) that

$$f(c) + \sum_{\xi \in D^{\tau+1}} \Psi^T(\xi) \cdot g(\xi, c(\xi), c(\xi^-)) \leq f(c_*).$$

Passing to the limit when T goes to infinite, we get the desired result (A.1):

$$\sum_{\xi \in D^{\tau+1}} f(\xi, c(\xi), c(\xi^-)) + \Psi(\xi) \cdot g(\xi, c(\xi), c(\xi^-)) \leq f(c_*).$$

Now assume that (d) is satisfied. Fix a node $\xi \in D$, there exist $\tau \geq t(\xi)$ and a finite process $\check{c} \in C^{\tau+1}(L)$ satisfying $g(\check{c}) \geq 0$, $f(\check{c}) \geq f^\tau(c_*)$ and $\check{c}\mathbf{1}_{[0,\tau]} = c_*\mathbf{1}_{[0,\tau]}$. Choosing $c = \check{c}$ in (A.1), it follows that

$$f^\tau(c_*) + \sum_{\xi \in D^{\tau+2}} \Psi(\xi) \cdot g(\xi, c_*(\xi), c_*(\xi^-)) \leq f(c_*).$$

Since

$$\lim_{\tau \rightarrow \infty} f^\tau(c_*) = f(c_*)$$

we get the desired result (A.2).

Now let c be a process in $C(L)$ with $g(c) \geq 0$ and such that (e) is satisfied, i.e., for any period $t \geq 1$, there exist $\tau \geq t$ and a finite process $\bar{c} \in C^{\tau+1}(L)$ satisfying $g(\bar{c}) \geq 0$, $f(\bar{c}) \geq f^\tau(c)$ and $\bar{c}\mathbf{1}_{[0,\tau]} = c\mathbf{1}_{[0,\tau]}$. Choosing $c = \bar{c}$ in (A.1), it follows that

$$f^\tau(c) + \sum_{\xi \in D^{\tau+2}} \Psi(\xi) \cdot g(\xi, c(\xi), c(\xi^-)) \leq f(c_*).$$

Therefore we can construct a strictly increasing sequence $(t_n)_{n \in \mathbb{N}}$ of integers such that

$$\lim_{n \rightarrow \infty} \sum_{\xi \in D^{t_n}} \{f(\xi, c(\xi), c(\xi^-)) + \Psi(\xi) \cdot g(\xi, c(\xi), c(\xi^-))\} \leq f(c_*)$$

and we get the desired result (A.3). \square

A.3 Proof of Proposition 5.2

We have to prove that for every period τ ,

$$\text{Pref}^i(p, a^i) \cap B^i(p, q, \kappa) \cap B^\tau = \emptyset.$$

Assume by way of contradiction that there exists a period τ , a plan $\bar{a} \in B^\tau$ in the budget set $B^i(p, q, \kappa)$ such that $\bar{a} \in \text{Pref}^i(p, a^i)$. Since a^i is physically feasible, we have $x^i(\xi) \leq \Omega(\xi)$ for each $\xi \in D$. It follows from Assumptions (A.2) and (A.3) that $U^i(x^i) \leq U^i(\Omega) < +\infty$, implying that $\lim_{T \rightarrow \infty} \Pi^{i,T}(p, a^i) = \Pi^i(p, a^i)$. In particular, if $\bar{a} \in \text{Pref}^i(p, a^i)$ then there exists $\varepsilon > 0$ such that $\Pi^i(p, \bar{a}) > \Pi^i(p, a^i) + \varepsilon$. Let F^i be the correspondence from $\Xi \times A$ to B^τ defined by $F^i(\pi, a) = B^i(\pi) \cap B^\tau \cap \psi^i(\pi, a)$ where ψ^i is the correspondence from $\Xi \times A$ to A^τ defined by³²

$$\forall \pi = (p, q, \kappa), \quad \forall a \in A, \quad \psi^i(\pi, a) = \{b \in A^\tau : \Pi^{i,\tau}(p, b) > \varepsilon + \Pi^i(p, a)\}.$$

Following the arguments in Páscoa and Seghir (2009), we can prove that the correspondence F^i is lower semi-continuous for product topologies on $\Xi \times A$. Recall that there exists a strictly increasing sequence $(T_n)_{n \in \mathbb{N}}$ with $T_n \in \mathbb{N}$ such that

$$\lim_{n \rightarrow \infty} ((p_n, q_n, \kappa_n), a_n^i) = ((p, q, \kappa), a^i)$$

³²Recall that if b belongs to A^τ , then $\Pi^i(p, b) = \Pi^{i,\tau}(p, b)$.

where

$$(p_n, q_n, \kappa_n) = (p^{T_n}, q^{T_n}, \kappa^{T_n}), \quad a_n^i = a^{i, T_n}.$$

Since F^i is lower semi-continuous and $\bar{a} \in F^i((p, q, \kappa), a^i)$, there exists v large enough and \bar{a}_v such that $\bar{a}_v \in F^i((p_v, q_v, \kappa_v), a_v^i)$ and $T_v \geq \tau$. In particular we have $\bar{a}_v \in B^i(p_v, q_v, \kappa_v) \cap B^\tau$ and

$$\Pi^{i, T_v}(p_v, \bar{a}_v) = \Pi^i(p_v, \bar{a}_v) > \Pi^i(p_v, a_v^i) + \varepsilon = \Pi^{i, T_v}(p_v, a_v^i) + \varepsilon.$$

This contradicts the optimality of a_v^i .³³

A.4 Proof of Proposition 5.3

Fix $\tau \in \mathcal{T}$ and let (π, \mathbf{a}) be a competitive equilibrium of the *truncated* economy \mathcal{E}^τ . Recall that $\pi = (p, q, \kappa)$ is a process of prices in Ξ and $\mathbf{a} = (a^i)_{i \in I}$ is an allocation with $a^i \in B^\tau$.

Applying standard arguments and following the notations of Section A.2, we can prove that for each agent i there exist,

- a family of non-negative Lagrange multipliers $(\gamma^i(\xi))_{\xi \in D}$ corresponding to the sequence of budget constraints (2.1) with $\gamma^i(\zeta) = 0$ when $t(\zeta) > \tau$;
- for each asset j , a family of Lagrange multipliers $(\rho^i(\xi, j))_{\xi \in D}$ corresponding to the sequence of minimum delivery constraints (2.3) with $\rho^i(\zeta, j) = 0$ when $t(\zeta) > \tau$;³⁴
- for each commodity ℓ , a family $(\chi^i(\xi, \ell))_{\xi \in D}$ of non-negative Lagrange multipliers corresponding to the sequence of collateral requirements (2.2) satisfying $\chi^i(\zeta, \ell) = 0$ when $t(\zeta) \geq \tau$;
- for each asset j , two families of non-negative Lagrange multipliers $(\alpha_\theta(\xi, j))_{\xi \in D}$ and $(\alpha_\varphi(\xi, j))_{\xi \in D}$ corresponding to the non-negative constraints on portfolio purchases and sales with $\alpha_\theta(\zeta, j) = \alpha_\varphi(\zeta, j) = 0$ when $t(\zeta) \geq \tau$;
- a family of super-gradients $(\nabla u^i(\xi, x^i(\xi)))_{\xi \in D^\tau}$ where $\nabla u^i(\xi)$ belongs to the super-differential $\partial u^i(\xi, x^i(\xi))$;
- for each asset j , a family of super-gradients $(\delta^i(\xi, j))_{\xi \in D^\tau \setminus \{\xi_0\}}$ where $\delta^i(\xi, j)$ is a super-gradient of $\Delta \mapsto [\Delta]^+$ at $\Delta^i(\xi, j) = V(p, \xi, j)\varphi^i(\xi^-, j) - d^i(\xi, j)$,

such that

(a) first order condition for consumption: for every $\xi \in D^\tau$,

$$\nabla u^i(\xi, x^i(\xi)) - \gamma^i(\xi)p(\xi) + \chi^i(\xi) + \sum_{\eta \in \xi^+} \gamma^i(\eta)p(\eta)Y(\eta) = 0; \quad (\text{A.5})$$

³³Recall that $((p_v, q_v, \kappa_v), \mathbf{a}_v)$ is a competitive equilibrium of the truncated economy \mathcal{E}^{T_v} .

³⁴We let $\rho^i(\xi_0, j) = 0$ since there is no delivery at initial node ξ_0 .

(b) first order condition for asset sales: for every $\xi \in D^{\tau-1}$ and every $j \in J$,

$$\begin{aligned} \gamma^i(\xi)q(\xi, j) - \sum_{\eta \in \xi^+} \lambda^i(\eta, j)\delta^i(\eta, j)V(p, \eta, j)/[p(\eta)v(\eta)] \\ - \chi^i(\xi)C(\xi, j) - \sum_{\eta \in \xi^+} \rho(\eta, j)D(p, \eta, j) + \alpha_\varphi(\xi, j) = 0; \end{aligned} \quad (\text{A.6})$$

(c) first order condition for deliveries: for every $\xi \in D^\tau \setminus \{\xi_0\}$ and every $j \in J$,

$$\lambda^i(\xi, j)\delta^i(\xi, j)\frac{1}{p(\xi)v(\xi)} - \gamma^i(\xi) + \rho^i(\xi, j) = 0. \quad (\text{A.7})$$

Observe that it follows from (A.5) that $\gamma^i(\xi) > 0$ for any node ξ . Multiplying (A.5) by $C(\xi, j)$, summing with (A.6) and using (A.7) we get that

$$\begin{aligned} \gamma^i(\xi)[q(\xi, j) - p(\xi)C(\xi, j)] - \sum_{\eta \in \xi^+} \lambda^i(\eta, j)\delta^i(\eta, j)\frac{V(p, \eta, j) - D(p, \eta, j)}{p(\eta)v(\eta)} \\ + \sum_{\eta \in \xi^+} \gamma^i(\eta)[p(\eta)Y(\eta)C(\xi, j) - D(p, \eta, j)] \\ + C(\xi, j)\nabla u^i(\xi, x^i(\xi)) + \alpha_\varphi(\xi, j) = 0. \end{aligned}$$

In particular, we get

$$\gamma^i(\xi)[p(\xi)C(\xi, j) - q(\xi, j)] \geq C(\xi, j)\nabla u^i(\xi, x^i(\xi)) - \sum_{\eta \in \xi^+} \lambda^i(\eta, j)\text{Def}(\eta, j).$$

Assume there exists a barrier period $t \in \text{Bar}$ satisfying $t < \tau$. We fix a node $\xi \in D^t$. Since the marginal utility for consuming the collateral exceeds the marginal default penalty, either (S.1) or (S.2) is satisfied. Assume first that (S.1) is satisfied. Since consumption markets clear there exists at least one agent i for which $x^i(\xi) \leq \widehat{\Omega}(\xi)$, implying that $p(\xi)C(\xi, j) - q(\xi, j) \geq 0$. The same conclusion follows if (S.2) is satisfied since for every agent i we have $x^i(\xi) \leq \Omega(\xi)$.

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